Comments on Algorithms for Grid Interfaces in Simulating Euler Flows\textsuperscript{1,2}

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Abstract: Some results concerning the algorithms for grid interfaces, which are crucial in simulating flows by zonal methods, are presented in this paper. It is indicated that the commonly used conservative interface scheme can ensure the discrete entropy condition, but it may be inconsistent and would bring a nonoverlapping solution on overlapping grids. A nonconservative interface matching obtained by interpolation can be monotonicity preserving, and it leads large conservation error when discontinuities are close to the interfaces. Methods for improvement of interface algorithms are also proposed.

Key words zonal method, grid interface, Euler equations

Introduction

Simulation of flows over complex geometries has become increasingly important in computational fluid dynamics. One of the key problems in the calculation is to choose interface schemes by which numerical approximations in different subregions are matched. To illustrate the problem, we consider Euler equations for 1D inviscid flows, namely,

\[ U_t + F(U) = 0. \]

As shown in Fig. a, grid for subdomain A and that for subdomain B are patched at \( x_I \), the grid interface. Thus the zonal method may be written as

\[ V_{n+1} = \begin{cases} V^n_I - \lambda_A (E^n_{i+1/2} - E^n_{i-1/2}), & i \leq I - 1, \\ V^n_I - \lambda_B (E^n_{i+1/2} - E^n_{i-1/2}), & i \geq I + 1, \\ Q, & i = I, \end{cases} \]

where \( \lambda = \Delta t/\Delta x \), \( \Delta t \) is time step, \( \Delta x \) is grid spacing, subscript or superscript A and B refer, respectively, to subdomain A and B. Here, schemes and grid spacings at two sides of \( x_I \) are not necessary the same. The above \( Q \) is the scheme for the grid interface, and it is either a conservative or a nonconservative treatment \textsuperscript{3}.

1. Conservative Algorithms

If we choose

\[ Q = V^n_I - \lambda_{AB} (E_I^{B}_{i+1/2} - E_I^{A}_{i-1/2}), \]

where \( \lambda_{AB} = 2\Delta t/((\Delta x_A + \Delta x_B)) \), then

\[ S^n = \Delta x_A \sum_{-\infty}^{I-1} V_I^n + \frac{\Delta x_A + \Delta x_B}{2} V_I^n + \Delta x_B \sum_{I+1}^{\infty} V_I^n = \text{const.} \]

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(3) is actually a typical conservative algorithm proposed in [2], [3], and [4], and it has been applied to practical simulations. Some comments concerning this conservative algorithm are stated as follows:

(a) Interface scheme (3) ensures a discrete entropy condition. It is a well known result that, if (1) is a scalar conservation law, on a uniform mesh a single consistent monotone scheme guarantees a discrete entropy condition (see [5]). By a proof similar to that for the result, it can also conclude that (2) satisfies the entropy condition if (3) is so chosen. Here, the conclusion is stated without a proof as

Proposition Let (1) be a scalar conservation law. If (3) is used at the interface and consistent monotone schemes are adopted at its two sides, then the discrete entropy condition holds.

(b) Algorithm (3) may be inconsistent with (1) and thus cause severe errors in or ruin the stability of (2). Let U be a solution of (1), consistency requires that

\[ U_I^{n+1} - Q = \Delta t. \]  

where \( \tau \to 0 \) as \( \Delta \to 0 \), \( \Delta = \max(\Delta t, \Delta x_A, \Delta x_B) \). Although \( E_{i+1/2}^A \) and \( E_{i+1/2}^B \) in (2) are consistent with \( F \) in (1), when schemes or grid spacings or both are different at two sides of \( x_I \), (5) may be violated, and (3) be inconsistent with (1). Numerical results show that this inconsistency affects numerical solution substantially. However, when (5) is satisfied, \( Q \) will lead no or only minor numerical errors. For more details, refer to [6].

(c) Numerical solutions do not overlap at the place grids overlap. Considering the case in Fig. b and using (3) at \( N \), one has

\[ V_N^{n+1} = V_N^n - \lambda_A (E_{i+1/2}^B - E_{i-1/2}^A). \]  

At \( L \), the corresponding point of \( N \) in subdomain B, (2) yields

\[ V_L^{n+1} = V_L^n - \lambda_B (E_{i+1/2}^B - E_{i-1/2}^A). \]  

Generally speaking, \( V_N^{n+1} \neq V_L^{n+1} \). For instance, let schemes in the two subdomains be different. but \( \lambda_A = \lambda_B = \lambda \) and \( V_N^n = V_L^n \), one has

\[ V_N^{n+1} - V_L^{n+1} = \lambda (E_{i-1/2}^A - E_{i-1/2}^B) \neq 0. \]  

Similar discussions can also be given at the left end of subdomain B. The differences between numerical solutions at the left and right end of subdomain B and A respectively would naturally give nonoverlapping solutions within the overlapping regions. This is shown by the solutions of a Riemann problem depicted in Fig. c, where the solid line stands for accurate solution, whereas the circles and diamonds give numerical solutions. Nonuniqueness related to this situation was proven in linear steady cases[7].

2. Nonconservative Algorithms

There are many nonconservative interface schemes. The most commonly used method is the interpolation approach, which is easy and always possible to implemented on chimera embedding grids. In Fig. a, a linear interpolation gives

\[ Q = (V_{I+1}^{n+1} - V_{I-1}^{n+1})(\Delta z_A + \Delta z_B)/\Delta z_A + V_{I-1}^{n+1} \]  

(9)
Another nonconservative matching used in practice is the reconstruction method [8]. Consider the grid point \( N \) in Fig. 2 b. The reconstruction procedure can be stated as: 1) reconstructing \( V \) between \( x_{N-1/2} \) and \( x_{N+1/2} \) on subdomain \( B \) as \( L(x, V_{n+1}^N) \); 2) giving solution at \( N \): 
\[
Q = V_N^n - L(x_{N+1/2}, V_{n+1}^N) / (\Delta x_A)
\]
If \( L(z, V_{n+1}^N) \) is piece-wise constant in subdomain \( B \), then it is obtained that
\[
Q = V_N^n - \lambda_A (E_{z+1/2} - E_{z-1/2}) .
\]

(10)

Based on above mentioned interpolation and reconstruction procedures, some conclusions can be drawn as follows:

(a) Compared with conservative methods, linear interpolation usually gives less oscillating solutions. For numerical examples, cf. [9]. Actually, we can prove the following results.

**Theorem** Suppose (1) is a scalar conservation law. Then

1) Let \( TV_c(V_{n+1}) \) be total variation of \( V_{n+1} \) obtained by (3), and \( TV^{nc}(V_{n+1}) \) be that of \( V_{n+1} \) obtained from (9), then, for a given \( V_{n} \),
\[
TV^{nc}(V_{n+1}) \leq TV_c(V_{n+1}).
\]

(11)

2) Let (2) be consisted of monotone schemes in both the subdomain \( A \) and \( B \), and (9) at the interface. Then, if \( V_{n} \) is strictly monotone, \( V_{n+1} \) is strictly monotone as well under the restriction
\[
\left| \frac{\lambda_A (E_{z+1/2}^A - E_{z-1/2}^A) - \lambda_B (E_{z+1/2}^B - E_{z-1/2}^B)}{V_{n+1} - V_{n}} \right| < 1.
\]

(12)

(b) Conservation error due to a nonconservative interface scheme appears primarily for the case where there is a discontinuity closed the interface. The interface solution \( Q \) given in (9) causes the conservation error between two time steps as
\[
S_{n+1}^n = (E_{z+1/2}^A - E_{z-1/2}^A) / (\Delta x_A)^2 \left( (\Delta x_A + \Delta x_B) \right)
+ \Delta t (E_{z+1/2}^B - E_{z-1/2}^B)
+ \Delta t (E_{z+1/2}^B - E_{z-1/2}^B) / (\Delta x_A + \Delta x_B)
\]

(13)

(cf. Fig. a). When there is no discontinuity near \( z_I \) and \( V_I - V_{I+1} = O(\Delta) \), it is known that
\[
S_{n+1}^n - S^n = \Delta \left( \Delta x_A \right).
\]

(14)

When \( (z_{I-2}, z_{I+2}) \) contains a discontinuity, (14) is no longer true. For example, the numerical flux of Lax-Friedrich scheme reads \( E_{z+1/2}^{LF} = (F_1 + F_{I+1})/2 + (V_I - V_{I+1})/2\lambda \). If a stationary shock is located in \( (z_{I-2}, z_{I-1}) \) and \( V_{I-2} - V_{I-1} = O(1) \), with the aid of Rankine-Hugoniot condition, it is known that on a uniform grid \( E_{z+1/2}^{LF} - E_{z-3/2}^{LF} = (V_{I-1} - V_{I-2})/(2\lambda) + O(\Delta) = O(1) \), which gives rise to
\[
S_{n+1}^n - S^n = O(\Delta).
\]

(15)

There,fore, the conservation error becomes one order larger in magnitude. It can be verified that this is also true if (10) is used that leads conservation error as
\[
S_{n+1}^n - S^n = \Delta t (E_{z+1/2}^A - E_{z-1/2}^B).
\]

(15)

As shown in Fig. d, the shock lags a little due to (9), while in Fig. e, it is almost not affected by (9).
3. Modification to Interface Algorithms

The above conservative and nonconservative algorithms have advantages and disadvantages as well, a combination or a little modification may make them to have a better performance.

(a) Self-adaptive algorithms: When there is a discontinuity near the interface a conservative algorithm is used; otherwise, a nonconservative one is adopted. Applying this algorithm, the solution for the problem given in Fig.c becomes much better (cf. Fig. f).

(b) Godunov scheme: Godunov scheme is employed at the interface. This can be done simply by replacing numerical fluxes at each side of the interface by the numerical fluxes of the Godunov scheme. The Godunov scheme is based on analytical solutions and it can be employed on nonuniform grids. Numerical results show that this method has a better stability than (3) (see [9]).

References
