A fully coupled method for simulation of wave-current-seabed systems

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1. Introduction

With the continued expansion of human activities along coastlines coupled with effects from the nature such as climate change, the need is imminent to improve our capability in studying and understanding multiphysics coastal processes, especially those at small-scales. Surface wave, current circulation, and seabed motion are three processes of special interest in view of their presence in various problems. Examples of such problems are seabed evolution, sediment deposition and resuspension, pollutant dispersion, coastal bridge collapse, and bay-river interaction. Nevertheless, these three processes are distinct in physics, and it is challenging to predict coastal hydrodynamic phenomena involved with their joint actions.

A number of efforts have been made in prediction of wave, current, and seabed morphology as well as their joint actions. Traditionally, the interaction among the three processes is simplified and considered in one-way or offline coupling, which takes action of a process on another into account but ignores the feedback on the former from the latter. For instance, in simulation of surface wave field, coastal current is usually used as a driving force and feedback from it is ignored, (e.g., [14]). Until recently, the most prevalent approach to simulate wave, current, and seafloor morphology is the so-called steady approach, in which the effect of seabed change on wave and current is assumed to be negligible [15]. However, it has been shown that the interaction among these processes may be significant (e.g., [16]). This interaction can be even more pronounced in extreme events such as tsunami due to earthquakes and storm surge associated with hurricanes [17]. Now it has become a trend to accurately take interaction among wave, current, and seabed into account in investigations of coastal
hydrodynamics, and a natural as well as feasible approach in this trend is to couple different models designed for individual processes [18–21]. In existing literature, however, most work deals with coupling between two of the three processes, such as coupling between current and wave or that between current and seabed. Recent investigations by Ferrarin et al. [22] and Tang et al. [23] are among the sparse efforts in studying interplay among wave, current, and seabed.

In prediction of phenomena in wave-current-seabed systems, such as storm surges traveling on sandy or muddy seabed in shallow water regions, certain features are desired for the numerical methods. In order to accurately model the phenomena, the numerical method should be robust and able to resolve interplay among the different processes and capture surfaces with steep slopes including shock-like discontinuities such as surge bores. However, most existing coastal ocean models such as POM and FVCOM do not have these desired features [24,25]. In correspondence with this situation, high-resolution schemes are introduced to simulate coastal flows involving wave, current, and seabed change. Here “high-resolution” refers to capturing shock-like waves with a few transition grid points and no artificial oscillations. For example, Rogers et al. [26] designed a Godunov-type method, together with an adaptive grid scheme, to simultaneously simulate ray-type wave–current interaction. Hudson and Sweby [27] made a systematical study of different formulations to couple current and seafloor and derived a few high-resolution schemes including a flux-limiter version of Roe scheme. An important issue in coupling current and seabed is the balance of the convection terms and the seabed slope terms in the shallow water equations [28].

The objective of this paper is to advance our capability in prediction of hydrodynamics involved with wave propagation, current flow, and seabed evolution. In particular, this paper explores a finite difference method for wave-current-seabed systems that is robust and capable to accurately capture phenomena involved with strong interplay among the three processes. The proposed method employs the wave action equation for surface wave, the shallow water equations for current flow, and the Exner equation for seabed evolution. In order to resolve water surfaces with steep slopes and other complicated phenomena, a high-resolution shock capturing scheme is used to construct the numerical method and enhance its robustness. The method presented in this paper is an extension of the method proposed by Hudson and Sweby [27,29] for current-seabed systems, and it is also a sequel to the coupling framework presented by Tang et al. [23]. It should be noted that simulation of a wave-current-seabed system is a challenge, and thus this research considers some simplified situations such as short and long waves and Cartesian grids. Nevertheless, it is a necessary step in developing our modeling capability and will provide foundations for simulation of more realistic problems.

2. Governing equations

As depicted in Fig. 1, the problem involves three types of physical processes, which are surface wave propagation, current circulation, and seabed evolution. Correspondingly, the governing equations consist of three components, and they are the wave action equation (e.g., [30]), the shallow water equations (e.g., [31]), and the Exner equation [32], which read as the following system:

\[
\frac{\partial W}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = f + g + r, \tag{1a}
\]

where

\[
W = \begin{pmatrix}
N \\
H \\
HU \\
HV \\
Hb
\end{pmatrix}, \quad F = \begin{pmatrix}
C_N N \\
H \frac{U^2}{2} + \frac{1}{2} g H^2 + g H b \\
H V \frac{U^2}{2} + \frac{1}{2} g H^2 + g H b \\
U^2 + V^2 \frac{n+1}{n}
\end{pmatrix}, \quad G = \begin{pmatrix}
C_N N \\
H V \\
H U V \\
H V^2 + \frac{1}{2} g H^2 + g H b \frac{U^2 + V^2}{n+1}
\end{pmatrix},
\]

Fig. 1. Schematic representation of wave-current-seabed systems.
In this system, \( t \) is the time, \( x \) and \( y \) are the Cartesian coordinates in the physical space, and \( \sigma \) and \( \theta \) are respectively the frequency and the angle in the spectrum space. \( N \) is the wave action, \( C_\sigma, C_y, C_x, \) and \( C_\rho \) are the wave speeds. \( S \) is the total source term, and it includes effects of wave growth resulting from wind, nonlinear transfer of wave energy through three-wave and four-wave interactions, and wave decay due to whitecapping, bottom friction, and depth-induced wave breaking [33]. \( H \) is the water depth, \( U \) and \( V \) are the depth-averaged velocities. \( \tau_\sigma^r \) and \( \tau_\rho^r \) are the bottom friction stresses, and \( \tau_\sigma^t \) and \( \tau_\rho^t \) are the wind shear stresses on the surface. \( S_{ux}, S_{uy}, S_{ux}, \) and \( S_{uy} \) are the radiation stresses. \( \rho \) is the water density, \( g \) is the gravity, and \( v_i \) is the turbulence eddy viscosity. \( H_b \) is the elevation of the seabed, \( \zeta \) is a coefficient, and \( \varepsilon = 1/(1 - \varepsilon) \), \( \varepsilon \) being the porosity of the seabed. \( P \) is a constant reflecting the effects of grain size and kinematic viscosity. \( m \) is a constant with the range of 1–4, and it is set as 3 in this study [34].

According to the linear wave theory (e.g., [33]), we have

\[
\begin{align*}
C_x = & \left( \frac{1}{2} + \frac{kH}{\sinh(2kH)} \right) \frac{\sigma k_x}{k^2} + U, \\
C_y = & \left( \frac{1}{2} + \frac{kH}{\sinh(2kH)} \right) \frac{\sigma k_y}{k^2} + V, \\
C_\sigma = & \frac{\sigma}{\partial \sigma} \left( \frac{\partial}{\partial x} + k_x \frac{\partial}{\partial m} \right) + \frac{1}{\rho} \left( \frac{\partial S_{ux}}{\partial x} + \frac{\partial S_{uy}}{\partial x} \right) \frac{\tau_\rho^t}{\rho} + \frac{\tau_\sigma^t}{\rho}, \\
C_\rho = & -k \left( \frac{\partial}{\partial m} + k_x \frac{\partial}{\partial m} \right), \\
\end{align*}
\]

and

\[
\begin{align*}
S_{ux} = & \rho g \int \frac{N}{\sigma} \left( \frac{1}{2} + \frac{kH}{\sinh(2kH)} \right) \cos^2 \theta + \frac{kH}{\sinh(2kH)} d\sigma d\theta, \\
S_{uy} = & \rho g \int \frac{N}{\sigma} \left( \frac{1}{2} + \frac{kH}{\sinh(2kH)} \right) \sin \theta \cos \theta d\sigma d\theta, \\
S_{uy} = & \rho g \int \frac{N}{\sigma} \left( \frac{1}{2} + \frac{kH}{\sinh(2kH)} \right) \sin^2 \theta + \frac{kH}{\sinh(2kH)} d\sigma d\theta.
\end{align*}
\]

Here, \( s \) is the space coordinate in the wave propagation direction, and \( m \) is a coordinate perpendicular to \( s \). Furthermore, \( k_x \) and \( k_y \) are wave numbers, and \( k = \sqrt{\kappa^2 + \kappa^2} \). In case of short wave and long wave, which are respectively defined as \( kH > \pi \) and \( kH < \pi/10 \), \( k \) being the wave number, it can be derived that

\[
\begin{align*}
C_x = & \left( 2 - n_w \right) \frac{g}{2\sigma} \cos \theta + (n_w - 1) \sqrt{gh} \cos \theta + U, \\
C_y = & \left( 2 - n_w \right) \frac{g}{2\sigma} \sin \theta + (n_w - 1) \sqrt{gh} \sin \theta + V, \\
C_\sigma = & (n_w - 1) \left( \frac{\sigma}{2} \left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) \right) - n_w \left( \frac{\sigma}{2} \left( \frac{\partial U}{\partial x} \cos^2 \theta + \frac{\partial V}{\partial y} \sin^2 \theta + \left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) \sin \theta \cos \theta \right) \right), \\
C_\rho = & (n_w - 1) \left( \frac{\sigma}{2} \left( \frac{\partial U}{\partial x} \sin \theta + \frac{\partial V}{\partial y} \cos \theta \right) \right) + \left( - \frac{\partial U}{\partial x} \cos^2 \theta + \frac{\partial V}{\partial y} \sin^2 \theta + \left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) \sin \theta \cos \theta \right), \\
\end{align*}
\]

and

\[
\begin{align*}
S_{ux} = & \frac{g}{2} \int \frac{N}{\sigma} (\cos^2 \theta + 1) d\sigma d\theta - \frac{g}{2} \int d\sigma d\theta, \\
S_{uy} = & \frac{g}{2} \int \frac{N}{\sigma} \cos \theta d\sigma d\theta, \\
S_{uy} = & \frac{g}{2} \int \frac{N}{\sigma} (\sin^2 \theta + 1) d\sigma d\theta - \frac{g}{2} \int N d\sigma d\theta.
\end{align*}
\]
Here, \( n_w = 1 \) and \( n_w = 2 \), corresponding to short and long wave, respectively. The bottom friction is evaluated as [35]

\[
\begin{align*}
\tau_x^b &= \rho g n_w H^{-1/3} U \sqrt{U^2 + V^2} + \frac{1}{2} \rho f_w \left( \frac{A \sigma}{\sinh(kH)} \right)^2, \\
\tau_y^b &= \rho g n_w H^{-1/3} V \sqrt{U^2 + V^2} + \frac{1}{2} \rho f_w \left( \frac{A \sigma}{\sinh(kH)} \right)^2,
\end{align*}
\]

which is simplified as [23]

\[
\begin{align*}
\tau_x^b &= \rho g n_w H^{-1/3} U \sqrt{U^2 + V^2} + (n_w - 1) \frac{f_w N}{\sigma H}, \\
\tau_y^b &= \rho g n_w H^{-1/3} V \sqrt{U^2 + V^2} + (n_w - 1) \frac{f_w N}{\sigma H},
\end{align*}
\]

in case of short and long wave. The turbulence eddy viscosity is calculated by [36]

\[ v_t = \alpha \rho g n_w H^{1/3} \sqrt{U^2 + V^2}, \]

where \( \alpha \) is the Manning coefficient, \( f_w \) is the wave friction factor, \( A \) is the wave amplitude and \( \sigma \) is a constant ranging from 0.3 to 1.

In system (1), the first equation is the wave action equation, the second, third, and fourth equations are the shallow water equations, and the fifth equation is the Exner equation. These equations represent models widely used in practical problems. The wave action equation is employed in models such as in SWAN [37]. The shallow water equations reproduce the framework for models such as SHORECIRC [38]. The Exner equation results from conservation of sediment mass, and it is also frequently used in engineering [39]. Since these governing equations are coupled with each other through the involving variables, even though the three types of physics are distinct and occur at different temporal and spatial scales, the multiphysics processes can be appropriately resolved only when they are solved in a strongly coupling way, in which the three equations, or all of the variables, are coupled with each other in each time step while marching in time.

System (1) is a generalization of current-seabed system used by many other authors. For instance, Liu et al. [34] developed a 2D model for morphology evolution using the shallow water equations and the Exner equation. Abderrazzaq and Paquier [40] test the capability of shallow water equations and the Exner equation in simulation of dam break flows. In addition, actually system (1) has been used in a few studies [22,23]. Since it is simple and can compare well with measurement data, the Exner equation is now used by more and more authors for morphodynamics study [34,40]. Nevertheless, its numerical computation is not trivial and needs special treatments [29,41].

The formulation of system (1) is a direct extension of the formulation recommended by Hudson and Swbey [29] for current-seabed systems; the former will reduce to the latter if the wave action equation and the term \( r \) are removed. They investigate several formulations with different treatments for \( f \) and \( g \) and suggest the one expressed in system (1) because of its better solution accuracy and numerical stability.

### 3. Numerical method

#### 3.1. Eigenvalues and eigenvectors

The system (1) can be rewritten as

\[
\frac{\partial \mathbf{W}}{\partial t} + A \frac{\partial \mathbf{W}}{\partial x} + B \frac{\partial \mathbf{W}}{\partial y} = \mathbf{f} + \mathbf{g} + \mathbf{r},
\]

where \( A = \partial \mathbf{f}/\partial \mathbf{W} \) and \( B = \partial \mathbf{g}/\partial \mathbf{W} \), being the Jacobian matrixes of \( \mathbf{f} \) and \( \mathbf{g} \), respectively, and

\[
A, B = \begin{pmatrix}
  a & b & N & n_x N \\
  0 & 0 & n_y N & 0 \\
  0 & n_x (g(H + H_b) - U^2) - n_y U V & 2 n_x U + n_y V & n_x U + n_y V \\
  0 & -n_y U V + n_x (g(H + H_b) - V^2) & n_y V & n_x U + n_y V \\
  0 & -n_x (U + V e) - n_y (V + U d) & n_d + n_y e & n_d + n_x c \\
  0 & 0 & 0 & 0
\end{pmatrix},
\]

in which

\[
\begin{align*}
a &= \left( 2 - n_w \right) \frac{g}{2 \sigma} + (n_w - 1) \sqrt{gH} \left( n_w \cos \theta + n_y \sin \theta \right) + (n_x U + n_y V), \\
b &= (n_w - 1) \left( \frac{2}{n_x} \frac{g}{\sqrt{H}} \right) \left( n_x \cos \theta + n_y \sin \theta \right) - \frac{N}{H} \cdot (n_x U + n_y V), \\
c &= \frac{p_z}{H} (U^2 + V^2) + 2 \frac{p_z}{H} (n_x U^2 + n_y V^2), \\
d &= 2 \frac{p_z}{H} U V.
\end{align*}
\]
Here short and long wave are considered. \((n_x, n_y)\) is a direction vector, \((n_x, n_y) = (1, 0)\) and \((n_x, n_y) = (0, 1)\), corresponding to \(A\) and \(B\), respectively.

Without the right hand side terms, system (1) is a hyperbolic system of conservation laws, and the eigenvalues of matrices \(A\) and \(B\) can be solved as

\[
\begin{align*}
\lambda_1 &= \alpha, \\
\lambda_2 &= 2\sqrt{Q}\cos\left(\frac{1}{3}\phi\right) + \frac{2}{3}(n_xU + n_yV), \\
\lambda_3 &= 2\sqrt{Q}\cos\left(\frac{1}{3}(\phi + 2\pi)\right) + \frac{2}{3}(n_xU + n_yV), \\
\lambda_4 &= 2\sqrt{Q}\cos\left(\frac{1}{3}(\phi + 4\pi)\right) + \frac{2}{3}(n_xU + n_yV), \\
\lambda_5 &= n_xU + n_yV,
\end{align*}
\]

\[
\begin{align*}
R &= -\frac{1}{2\gamma}(n_xU^3 + n_yV^3) + \frac{g}{6}(n_xU + n_yV)(2H + 2H_b - Hc), \\
Q &= -\frac{1}{9}(n_xU^2 + n_yV^2) - \frac{g}{3}(H + H_b + Hc), \\
\cos(\phi) &= R/\sqrt{-Q^2}.
\end{align*}
\]

More discussion on the eigenvalues can be found in Spiegel and Liu [42]. Corresponding to the eigenvalues, the eigenvectors are derived as

\[
\begin{align*}
e_1 &= (1, 0, 0, 0, 0)^T, \\
e_k &= \left(1, n_x(s_k^x + n_s^y), n_xc_k + n_yd_k, n_xV + n_yd_k, \frac{\mu_k}{gH}\right)^T, \\
e_5 &= \left(0, 1, n_xU + n_yW, n_xW + n_yV, -\frac{(H + H_b)}{H}\right)^T, \quad d \neq 0, \\
e_5 &= \left(0, 0, n_y, n_x, 0\right)^T, \quad d = 0,
\end{align*}
\]

where \(k = 2, 3, 4,\) and

\[
\begin{align*}
\mu_k &= (n_xU^2 + n_yV^2) - g(H + H_b) + \lambda_k(\lambda_k - 2(n_xU + n_yV)), \\
w &= (n_xV + n_yU) - \frac{(H + H_b)}{Hc}(n_xU + n_yV), \\
s_k^x &= (2 - n_y)\frac{N(U - \lambda_k)}{H(\frac{\gamma}{2}\cos\theta + U - \lambda_k)} + (n_y - 1)\frac{N(U - \lambda_k) - 0.5\sqrt{\gamma H} \cos \theta}{H(\sqrt{\gamma H} \cos \theta + U - \lambda_k)}, \\
s_k^y &= (2 - n_y)\frac{N(V - \lambda_k)}{H(\frac{\gamma}{2}\sin\theta + V - \lambda_k)} + (n_y - 1)\frac{N(V - \lambda_k) - 0.5\sqrt{\gamma H} \sin \theta}{H(\sqrt{\gamma H} \sin \theta + V - \lambda_k)}, \\
e &= (1 - n_y)\frac{N}{2H}.
\end{align*}
\]

Notice that \(e_5\) is not continuous at \(d = 0\).

### 3.2. Discretization

In this paper, system (3) will be solved using a finite difference method on Cartesian coordinates. An operator splitting method is applied to discretize system (1), which has two steps that read as

Step 1:

\[
W_{ij}^{n+1} = W_{ij}^n - s_x\left(F_{i+\frac{1}{2},j}^n - F_{i-\frac{1}{2},j}^n\right) - s_y\left(G_{i,j+\frac{1}{2}}^n - G_{i,j-\frac{1}{2}}^n\right) + s_x\tilde{F}_{i+\frac{1}{2},j} + s_y\tilde{G}_{i,j}.
\]

Step 2:

\[
W_{ij}^{n+1} = W_{ij}^n + \tilde{r}.
\]

Here \(s_x = \Delta t/\Delta x\), and \(s_y = \Delta t/\Delta y\). \(\Delta t\) being the time step, and \(\Delta x\) and \(\Delta y\) being the grid spacing. In the discretization, first the solution marches from time level \(n\) to an intermediate level \(n^*\), and then it proceeds to time level \(n + 1\). \(F_{i+\frac{1}{2},j}^n, G_{i,j+\frac{1}{2}}^n, \tilde{F}_{i+\frac{1}{2},j}, \tilde{G}_{i,j}\) and \(\tilde{r}\) are respectively the approximations for operators \(F, G, f, g\), and \(r\) defined in Eq. (1), and their constructions will be discussed as follows. Step 1 is an advection step, and step resembles a source term integration step.
In Step 1, the numerical fluxes $F_{i+1/2}^j$, $G_{j+1/2}$ are evaluated using techniques developed for hyperbolic systems of conservation laws. In this paper, following the idea of Boris and Book [43] and van Leer [44], we design them as combinations of numerical fluxes of a first-order and a second-order scheme. The numerical flux in $x$-direction is determined as

$$F_{i+1/2} = F_{i+1/2}^0 + \Phi \left( F_{i+1/2}^0 - F_{i-1/2}^0 \right), \quad (8a)$$

where $F_{i+1/2}^0$ and $F_{i-1/2}^0$ are numerical fluxes of a first-order and a second-order scheme, respectively, $\Phi$ is a flux limiter, and $\Phi = \text{diag}(\Phi(\theta_k))$. Here $\theta_k$ is a monitor for solution smoothness, and $\Phi(\theta_k)$ is the widely used Minmod function:

$$\Phi(\theta_k) = \max(0, \min(1, \theta_k)). \quad (8b)$$

Candidates for other flux limiters are available (e.g., [45]). Using the numerical flux of a first-order upwind scheme and the second-order Lax–Wendroff (L–W) scheme, Eq. (8a) can be expressed as

$$F_{i+1/2} = \frac{1}{2} (F_{i+1/2} + F_{i,j}) - \frac{1}{2} \sum_{k=1}^{k_1} (\alpha_k |\lambda_k| (1 - \Phi(\theta_k)) |\lambda_k|) c_{i,k} + \Phi(\theta_k) (1 - |\lambda_k|) c_{i,k} + \Phi(\theta_k) G_{i,j} + A_{i,j}(G_{i,j+1} - G_{i,j-1}). \quad (9a)$$

Similarly, the numerical flux in $y$-direction is derived as

$$G_{j+1/2} = \frac{1}{2} (G_{j+1/2} + G_{i,j}) - \frac{1}{2} \sum_{k=1}^{k_1} (\alpha_k |\lambda_k| (1 - \Phi(\theta_k)) |\lambda_k|) c_{j,k} + \Phi(\theta_k) (1 - |\lambda_k|) c_{j,k} + \Phi(\theta_k) F_{i,j} + A_{i,j}(F_{i,j+1} - F_{i,j-1}). \quad (9b)$$

In expressions for numerical fluxes (9a) and (9b),

$$\theta_k = (\lambda_k)_{i,k+1/j} / (\lambda_k)_{i,k-j}, \quad I = i - \text{sgn}(s_i \lambda_k)_{i,k-j}, \quad (10a)$$

and

$$\theta_k = (\lambda_k)_{i+j,k} / (\lambda_k)_{i-k,j}, \quad J = j - \text{sgn}(s_j \lambda_k)_{i,k-j}, \quad (10b)$$

where $\alpha_k$ is evaluated as

$$\begin{align*}
\alpha_1 &= \Delta(N) - \sum_{m=2}^e e_m \alpha_m, \\
\alpha_k &= (w - V)(\lambda_k - \lambda_{k1})(\lambda_k - \lambda_{k2}) + (w - U)(\lambda_k - \lambda_{k1})(\lambda_k - \lambda_{k2}), \\
\alpha_5 &= n_\psi \frac{\Delta(\Delta(HV) - V\Delta(H))}{w - V} + n_\psi \frac{\Delta(\Delta(UH) - U\Delta(H))}{w - U},
\end{align*} \quad (10c)$$

where $k, k_1, k_2 = 2, 3, 4$, and $k_1$ and $k_2$ are distinct from each other, $e \neq 0$, and $e_m$ is the first element in $e_m$, and

$$\begin{align*}
\psi_{\text{xf}} &= ((2U - \lambda_{k1} - \lambda_{k2})UV - (H + H_3)gV + (\lambda_{k1}\lambda_{k2} + g(H + H_3) - U^2)w)\Delta(H) + (w - V)(ghH\Delta(H)) + (2U - \lambda_{k1} - \lambda_{k2})\Delta(UH), \\
\psi_{\text{yg}} &= ((2V - \lambda_{k1} - \lambda_{k2})UV - (H + H_3)gU + (\lambda_{k1}\lambda_{k2} + g(H + H_3) - V^2)w)\Delta(H) + (w - U)(ghH\Delta(H)) + (2V - \lambda_{k1} - \lambda_{k2})\Delta(UH),
\end{align*} \quad (10d)$$

where $\Delta = (\lambda_{i+1,j} - \lambda_{ij})$ when $(n_x, n_y) = (1, 0)$, and $\Delta = (\lambda_{ij+1} - \lambda_{ij})$ when $(n_x, n_y) = (0, 1)$. When $d = 0$, one has

$$\begin{align*}
\alpha_1 &= \Delta(N) - \sum_{m=2}^e e_m \alpha_m, \\
\alpha_k &= (\lambda_k - \lambda_{k1})(\lambda_k - \lambda_{k2}) + (\lambda_k - \lambda_{k1})(\lambda_k - \lambda_{k2}), \\
\alpha_5 &= n_\psi \frac{\Delta(\Delta(HV) - V\Delta(H))}{w - V} + n_\psi \frac{\Delta(\Delta(UH) - U\Delta(H))}{w - U},
\end{align*} \quad (10e)$$

and

$$\begin{align*}
\psi_{\text{xf}} &= ((\lambda_{k1}\lambda_{k2} + g(H + H_3) - U^2)\Delta(H) + ghH\Delta(H)) + (2U - \lambda_{k1} - \lambda_{k2})\Delta(UH), \\
\psi_{\text{yg}} &= ((\lambda_{k1}\lambda_{k2} + g(H + H_3) - V^2)\Delta(H) + ghH\Delta(H)) + (2V - \lambda_{k1} - \lambda_{k2})\Delta(UH),
\end{align*} \quad (10f)$$

$F_{ij}$ and $G_{ij}$ will be determined in such way that they balance the numerical fluxes $F_{i+1/2}$ and $G_{j+1/2}$, satisfying the C-property [28]. As a result, $F_{ij}$ is comprised by two parts:

$$F_{ij} = f_{i+1/2} + f_{i-1/2}, \quad (11a)$$

and

$$G_{ij} = g_{j+1/2} + g_{j-1/2}, \quad (11b)$$
The two terms in the right hand side of Eqs. (11a) and (11b) are constructed by combining a first-order and a second-order approximation of \( f \) and \( g \), similarly as shown in Eq. (8a). The resulting expression of \( f_{i,j}^m \) is derived as

\[
f_{i,j}^m = \frac{1}{2} \sum_{k=1}^{4} (\beta_k c_k (1 \pm \sgn(\lambda_k)(1 - \Phi(\theta_k^m)(1 - |s_k \lambda_k|))))_{i,j} + \frac{s_k}{2} \Phi(\theta_k^m)(A_g)_{i,j+1} + (A_g)_{i,j}.
\]

By the same procedure, the expression of \( g_{i,j}^m \) is given as

\[
g_{i,j}^m = \frac{1}{2} \sum_{k=1}^{4} (\beta_k c_k (1 \pm \sgn(\lambda_k)(1 - \Phi(\theta_k^m)(1 - |s_k \lambda_k|))))_{i,j} + \frac{s_k}{2} \Phi(\theta_k^m)(B_f)_{i,j+1} + (B_f)_{i,j}.
\]

In Eqs. (12a) and (12b), \( \beta_k \) reads as

\[
\begin{align*}
\beta_1 &= -\sum_{k=2}^{4} \beta_k, \\
\beta_k &= n_s (2U - \lambda_k - \lambda_{k+1})/2 \Delta H \Delta (\lambda_k) + n_s (2V - \lambda_k - \lambda_{k+1})/2 \Delta H \Delta (\lambda_k), \\
\beta_5 &= 0,
\end{align*}
\]

where \( k = 2, 3, 4, \) and \( k_1 \) and \( k_2 \) are different from each other.

Step 1 is a conservative scheme, and it couples wave, current, and seafloor when the numerical solution advances from time step \( n \) to the intermediate level. For instance, as shown in Eqs. (10c) and (10e), since \( \tau_j \) is related to \( \tau_{j-1}, \tau_{j}, \tau_{j+1}, \) and \( \tau_{j+2} \), which are determined by variables of the current and seabed, it is known that the update of wave action is directly coupled with the updates of the current and seabed during the marching in time. The expressions of numerical fluxes (9) and seabed terms (11) are generalizations of those formulated for current-seabed systems by Hudson and Sweby [27,29]. This step will reduce to an upwind scheme and the L–W scheme when \( \Phi(\theta_1) = 0 \) and \( \Phi(\theta_2) = 1 \), respectively. In addition, the proposed method could be further enhanced by evaluating the fluxes (9) and seabed terms (11) with the Roe averages. However, this increases the computation load substantially but, as indicated by Hudson and Sweby [27,29], presents a very similar solution, and thus it is not pursued in this paper.

In designing Step 2, the wave action equation is solved by the following scheme:

\[
N_{i,j,k}^{n+1} = \frac{1}{2} \sum_{j \neq j_0-1} N_{i,j',k}^{n+1} - \Delta D_i (C_s N_{i,j,k}) - \Delta D_i (C_N N_{i,j,k}),
\]

where \( D_m \) is the central difference operator with respect to \( m \)-direction \((m = i,j,k,l)\). For instance, \( D_k(i,j,k,l) = (i_{k,j,k+l} - i_{k,j,k-1})/(2\Delta s) \). For the shallow water equations, a MacCormack-type scheme is used [46]:

\[
\begin{align*}
&H_{i,j}^{n+1} = H_{i,j}^n + 2 \Delta t D_i \left( (vH)_{i,j}^n (V)_{i,j}^n + \Delta t D_i \left( (vH)_{i,j}^n (D_i (V)_{i,j}^n) - \frac{\Delta t}{\rho} \left( \frac{\tau_i - \tau_j}{\rho} \right)_{i,j} \right) ight), \\
&\frac{(HV)_{i,j}^n}{(U)_{i,j}^n} = (HV)_{i,j}^n + 2 \Delta t D_i \left( (vH)_{i,j}^n (D_i (U)_{i,j}^n) + \Delta t D_i \left( (vH)_{i,j}^n (D_i (U)_{i,j}^n) - \frac{\Delta t}{\rho} \left( \frac{\tau_i - \tau_j}{\rho} \right)_{i,j} \right) \right),
\end{align*}
\]

and

\[
\begin{align*}
&H_{i,j}^{n+1} = \frac{1}{2} (H_{i,j}^n + H_{i,j}^n), \\
&(HV)_{i,j}^{n+1} = \frac{(HV)_{i,j}^n + (HV)_{i,j}^n}{2} + \Delta t D_i \left( (vH)_{i,j}^n (D_i (V)_{i,j}^n) + \frac{\Delta t}{\rho} \left( \frac{\tau_i - \tau_j}{\rho} \right)_{i,j} \right), \\
&(HV)_{i,j}^{n+1} = \frac{(HV)_{i,j}^n + (HV)_{i,j}^n}{2} + \Delta t D_i \left( (vH)_{i,j}^n (D_i (U)_{i,j}^n) + \frac{\Delta t}{\rho} \left( \frac{\tau_i - \tau_j}{\rho} \right)_{i,j} \right),
\end{align*}
\]

where \( D_m \) and \( D_m \) \((m = i,j)\) are backward and forward difference, respectively. For example, \( D_i(i,j) = (i_{j,j} - i_{j_{i-1}})/(2\Delta x) \). For the seabed morphology,

\[
(H_b)_{i,j}^{n+1} = (H_b)_{i,j}^n,
\]
After Step 1 and Step 2 are finished, the whole system of wave, current, and seabed morphology has marched from time level \( n \) to \( n + 1 \). The time step is restricted by

\[
\Delta t = \min \left\{ \frac{CFL \cdot \Delta x}{\max(\|A\|)}, \frac{CFL \cdot \Delta y}{\max(\|B\|)}, \frac{CFL \cdot \Delta \sigma}{C_\sigma}, \frac{CFL \cdot \Delta \theta}{C_\omega}, \frac{\text{von} \cdot \min(\Delta x^2, \Delta y^2)}{v_i} \right\},
\]

where CFL and von are prescribed parameters.

4. Numerical experiments

4.1. Wave, current, and seabed morphology

The numerical method developed in the last section is validated using test problems with analytical solutions. The first test problem is a wave propagation problem. In this problem, the current field is prescribed as

\[
U = 1 \text{ m/s}, \quad V = 1 \text{ m/s}, \quad H = 2 \text{ m},
\]

and the wave action equation has the following source term:

\[
S = 4\alpha N \left( (x - t)^2 + (y - t)^2 - t^2 \right) \left( x + y - 2t - a_1 t - C_x (x - t) - C_y (y - t) \right) + a_2 N \left( \frac{1}{1 - \exp(-a_6 t)} + \frac{1}{\sigma} - 2a_2 (\sigma - a_3) \right) + \frac{a_4 N C_\omega \cos a_4 t}{2 + \sin a_4 \theta}.
\]

![Fig. 2. Solutions for wave propagation in physical space at \( t = 3.5 \text{ s} \). (a) Numerical solution, (b) analytical solution, and (c) numerical and analytical solutions along cross section \( a-a \).](image-url)
The problem is solved by

\[ N = \frac{1}{\sigma_0} \exp\left(-\frac{\sigma_0}{a_6} t \right) \left( 2 + \sin a_4 \theta \right) \exp \left( -a_2 (\sigma - \sigma_3)^2 \right) \times \exp \left( -a_5 ((x - t)^2 + (y - t)^2 - \alpha_1 r^2) \right). \]  

(18c)

In the computation, \( a_1 = 1, \ a_2 = 0.01, \ a_3 = 0.5, \ a_4 = 1, \ a_5 = 0.005, \ a_6 = 0.005, \) and \( a_7 = 0.3. \) Besides, \( \Delta x = \Delta y = 0.4 \text{ m}, \ \Delta \sigma = 0.25 \text{ s}^{-1}, \ \Delta \theta = \pi/25 \text{ rad}, \) and the CFL number is 0.5. The numerical solution in presence of short wave, together with the exact solution and a solution obtained in [23], is shown in Fig. 2.

The second test is a 1-D Riemann problem of an inviscid flow with the following initial discontinuity:

\[ u(x, 0) = \begin{cases} u_1, & x < 0, \\ u_2, & x \geq 0. \end{cases} \]

\[ \sigma(x, 0) = \begin{cases} \sigma_1, & x < 0, \\ \sigma_2, & x \geq 0. \end{cases} \]

\[ \theta(x, 0) = \begin{cases} \theta_1, & x < 0, \\ \theta_2, & x \geq 0. \end{cases} \]
\[ U = 0 \text{ m/s, } H = 5 \text{ m, } x < 0; \quad U = 0 \text{ m/s, } H = 3 \text{ m, } x > 0. \]  
\[ \Delta x = 0.2 \text{ m, and the CFL number is 0.9. The exact solution of the problem can be obtained in literature (e.g., [47]). In comparison with a MacCormark-type scheme in [23], the numerical method proposed in this paper performs better in view that it captures the surge front with high-resolution and resolves the expansion wave with a good accuracy (Fig. 3).} \]

The third test is a 1-D seabed evolution problem, which has the following initial seafloor condition:

\[ H_b = 1 + \cos \left( \frac{\pi x}{10} \right). \]  
\[ H = (3 - H_b), \quad U = \frac{1}{H^2}. \]

Fig. 6. Temporal variation of water surface (the upper branch) and bed elevation (the lower branch) at three locations, obtained by experiment [49], the method of Tang et al. [23], and the method of this paper. \( t_0 = \sqrt{H_0/g} \) (a) \( x = 0.25 \) m, (b) \( x = 0 \) m, and (c) \( x = 0.25 \) m.
The problem is a test example used by others (e.g., [48]) and it is solved by
\[ H_b = 1 + \cos \left( \frac{\pi}{10} \left( x - \frac{t}{(3.0 - H_b)^\xi} \right) \right), \]
\[ \text{(20c)} \]

The grid spacing is chosen as \( \Delta x = 0.2 \) m, and the CFL number is 0.95. As seen in Fig. 4, the numerical solution agrees with the analytical solution, presenting a sharp profile at the peak of the seabed elevation, rather than spurious oscillations associated with other conventional schemes such as a MacCormack-type scheme [23,46].

4.2. Dam-break flow over a movable bed

A laboratory experiment has been made for dam-break flow over a movable, saturated, and initially flat bed (Fig. 5). The problem is formulated as the following initial value problem:
\[
\begin{aligned}
U &= 0 \text{ m/s}, & H &= 0.1 \text{ m}, & H_b &= 0 \text{ m}, & x < 0, \\
U &= 0 \text{ m/s}, & H &= 0.0 \text{ m}, & H_b &= 0 \text{ m}, & x > 0.
\end{aligned}
\]
\[ \text{(21)} \]

In the computation, \( P = 0.003, \xi = 1.25, \text{ and } n = 0.025. \Delta x = 0.05 \text{ m}, \text{ CFL} = 0.95, \text{ and } \text{von} = 0.5. \]

Fig. 6 shows the numerical solution for evolution of water surface and bed elevation at three locations. From the figure it is seen that there is a good agreement between the results obtained with the proposed method and the experiment data. Furthermore, in comparison with the method of [23], which is a MacCormack-type scheme, the method developed in this paper captures more details observed in the experiment at the dam location shortly after the removal of the dam, primarily during \( 0 < t/t_0 < 6 \), when there is strong interaction between the current flow and the channel bed (Fig. 6b). This is attributed to the fact that the latter couples the current and seabed morphology more closely than the former. An instantaneous solution for water surface and channel bed elevation is presented in Fig. 7. This figure indicates that the method of Tang et al. has many oscillations for both water surface elevation and channel bed, while the method of this paper does not. Interestingly, both of the methods capture a seafloor wave front at \( x = 1.7 \) m (Fig. 7b).

4.3. Wave-driven flow over a sand dune

In this problem, a wave with increasing strength drives an initially static water body over a sand dune, and there is a strong interaction among wave, current, and sand dune. In order to demonstrate its capabilities and features, the numerical solution obtained with the method proposed in this paper is compared with those by the method of [23], an upwind scheme, and the L–W scheme. The upwind scheme and the L–W scheme are obtained by respectively setting \( \Phi(\theta_k) = 0 \) and \( \Phi(\theta_k) = 1 \) in Eqs. (9) and (12). Initially, the morphology of the sand dune is
\[ H_b = \exp \left( -0.5 \times (x^2 + y^2) \right), \]
\[ \text{(22a)} \]
the current field is
\[ U = 0 \text{ m/s}, \quad V = 0 \text{ m/s}, \quad H = (2.0 - H_b) \text{ m}, \]
\[ \text{(22b)} \]
and the initial wave action distribution is
\[ N = 1 \text{ kg/s^3}. \]
\[ \text{(22c)} \]
Fig. 8. Computed seabed morphology at $t = 1200$ s. (a) Method of this paper, (b) method of Tang et al. [23], (c) an upwind scheme, and (d) the L–W scheme.

Fig. 9. Water surface elevation at $t = 1200$ s. (a) Method of this paper, (b) method of Tang et al. [23], (c) an upwind scheme, and (d) the L–W scheme.
Fig. 10. Magnitude of velocity at $t = 1200$ s. (a) Method of this paper, (b) method of Tang et al. [23], (c) an upwind scheme, and (d) the L–W scheme.

Fig. 11. Wave action in physical space at $t = 1200$ s. (a) Method of this paper, (b) method of Tang et al. [23], (c) an upwind scheme, and (d) the L–W scheme.
At the upstream,
\[ H = 2 \text{ m}, \quad x = -10 \text{ m}, \]
and a wind is blowing downstream, which is modeled by the source term:
\[ S = \frac{-\gamma g (11.75 \times (\tanh(x - 0.01t + 20) - 1)) \sigma}{b^2 \pi} E(\sigma, \theta), \]
where \( E \) is the wave density and it equals \( N/\sigma \). \( S \) given in (22e) reflects the combined effects from wave breaking, whitecapping, etc. In the computation, \( \gamma g = 1 \), \( \beta = 0.5 \), \( \Delta x = \Delta y = 0.4 \text{ m} \), \( \Delta \sigma = 0.1 \text{ s}^{-1} \), \( \Delta \theta = 0.775 \text{ rad} \), CFL = 0.1, and \( \text{von}=0.05 \). Again, \( P = 0.003 \), \( \zeta = 1.25 \), and \( n = 0.025 \).

The computed results are presented in figures from Fig. 8 to Fig. 12. Under the action of the surface wave, the sand dune moves downstream, and its height gradually decreases. The initial circular sand dune evolves into an asteroid, and low-elevation regions are forming at lateral sides and the downstream side of the sand dune (Fig. 8). Additionally, the solutions for wave, current, and dune morphology have structures and traces of the others, which is a clear indication that these three components affect each other and there is strong interaction among them (Figs. 8–11). It is seen that all of the four methods produce similar solutions, which are consistent with those of previous studies [27]. However, indeed there is difference in performance of the four different numerical methods. From these figures, we can see that the classic second-order L–W method produces many oscillations that are apparently artificial, such as those at the upstream end in Fig. 9d and 10d. Notwithstanding a very similar solution, the method of Tang et al. presents fewer oscillations than the L–W scheme. In contrast, the first-order upwind method produces a smooth but severely smeared solution. Furthermore, it presents almost no trace of the sand dune, and this illustrates clearly that the smearing is so strong that basically it shows little reaction of the wave field to the current and seabed. Nevertheless, the method of this paper not only

Fig. 12. Wave action in spectral space at \( t = 1200 \text{ s} \). (a) Method of this paper, (b) method of Tang et al. [23], (c) an upwind scheme, and (d) the L–W scheme.
produces no spurious oscillations, as the upwind scheme does, but also resolves flow structures with resolution similar to that by the L–W method and the method of Tang et al. In region 4 < x < 5 m, it shows a single peak for seabed elevation, current elevation and speed, and wave action, while the method of Tang et al. and the L–W scheme present multiple peaks there (Figs. 8–11). These figures demonstrate that the proposed scheme has the best performance among the four methods.

5. Conclusions
A fully coupled method has been developed for prediction of phenomena in wave-current-seabed systems. In order to test its capability and performance, the developed method is applied in different flows and compared with analytical, experimental, and other numerical approaches. It is demonstrated that the newly developed method is able to simulate joint actions of wave, current, and seafloor. Indeed the proposed method is robust in view that it retains capability of first-order upwind schemes in suppressing artificial oscillations, and also it has the accuracy of second-order schemes in resolving flow structures.

In order to better evaluate the proposed method, more analytical as well as numerical studies are necessary on its solution accuracy, stability, and convergence. This paper focuses on Step 1 in the splitting method, its combination with Step 2 may not lead to overall second-order accuracy in time, and thus an improvement on design of Step 2 is needed to make the whole scheme to fully achieve second-order accuracy. The time step of the method is restricted since explicit schemes are used in the two steps, and the restriction could be significantly relaxed by developing an implicit version of the scheme. The method is more expensive than traditional methods, and its efficiency improvement in aspect of CPU time and computer storage remains as a topic. Furthermore, the proposed method only considers short and long waves, and its application will be much broader if it can be extended to full range of wave spectrum. Given its attracting features illustrated in this paper, these issues shall be considered for our future study.

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