ON NONCONSERVATIVE ALGORITHMS FOR GRID INTERFACES∗

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Abstract. In computations of fluid flows by domain decomposition methods, the necessity of conservation at grid interfaces is now widely claimed. In this paper we consider nonconservative algorithms for the interfaces. Our investigation begins with discussions about typical nonconservative interface treatments for one-dimensional calculations. The analysis shows that the conservation error of a numerical solution caused by a nonconservative interface matching has an upper bound when the solution itself is bounded. Furthermore, if the numerical solution converges as the mesh size goes to zero, it converges to a weak solution of the problem under certain conditions that may be detected numerically. Also, we acquire similar results for two-dimensional calculations on grids intersecting with each other in an arbitrary way. In order to illustrate the theoretical results, we present numerical examples and demonstrate that, under those conditions, conservation error reduces and accuracy for jumps as well as locations of discontinuities improves as the mesh size decreases.

Key words. grid interface, nonconservative interface algorithm, conservation error, weak solution

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1. Introduction. Grid interfaces, which divide the region of interest into many subregions, are becoming more common in flow simulations for three reasons. First, in computing a flow over a complex geometry such as realistic aircraft configuration, it would be extremely difficult to generate a single smooth grid that wraps itself around the entire body. To simplify this problem, one can use several grids at once. Thus, conditions are needed at grid interfaces that are interior to the problem domain. Second, interface problems arise from limitations of computer speed and storage, which preclude the possibility of solving the unsteady, three-dimensional, Reynolds-averaged, Navier–Stokes equations in all flow regions. Since viscous effects usually dominate only in certain regions, Euler equations that require less CPU time and computer storage may be employed in the other regions. Finally, the interfaces may come from the requirement of different numerical schemes for different flow structures. For instance, both vortexes and shock waves, whose structures are distinct from each other, may be present in a flowfield, and their accurate simulation dictates different numerical formulations.

Conditions at grid interfaces, by which numerical solutions of different subdomains are connected with each other, are crucial in calculations by domain decomposition methods. A good interface treatment should be consistent with governing equations and not result in a divergent computation or a wrong solution. It was believed that conservative interface algorithms were necessary for the correct shock location and jump for shocks passing through grid interfaces. Accordingly, Warming and Beam [1] derived transition operators for switching conservatively between an upwind...
scheme and the MacCormack scheme. Rai [2] proposed a common grid line and devised a conservative matching procedure on the line. Berger [3] examined conservative treatments theoretically and stated that such treatments guaranteed that if numerical solutions converged, they converged to weak solutions of the approximated partial differential equations (PDEs). Recently, Pärт-Enander and Sjögreen [4] and Wang [5] successfully developed certain conservative procedures for practical calculations. Now, conservative interface treatments get more and more attention. Nevertheless, there are some negative aspects about them. Pärт-Enander and Sjögreen [4] proved that a conservative interface scheme of flux-interpolation type could only be weakly stable and thus lead to numerical instability. Wu [6] made a thorough investigation and proved that in steady linear cases it would give nonunique solutions at locations where grids overlapped. Tang and Lee [7] found that a conservative matching based on flux-interpolation was usually inconsistent with the PDE and induced severe numerical errors. Tang and Lee [8] also showed it would give nonoverlapping solutions on overlapping grids in calculations of systems of conservation laws. Worst of all, it would be difficult, if not impossible, to realize conservation at interfaces between multidimensional Chimera embedding grids that are often encountered in engineering.

Karni [9] indicated that nonconservative formulations for shock relations were not uniquely determined by flow states at either side of a shock. Hou and Le Floch [10] showed that such a difference scheme might converge to the solution of another inhomogeneous conservation law. Although schemes in all subdomains may be in conservative forms, numerical results are not conservative globally when grid interfaces are not treated conservatively. Computations demonstrated that this global nonconservation would lead to inaccurate shock jumps and locations, especially when the interfaces are close to the exact shock locations; cf. [11]. As pointed out by Wang [5], the effects of nonconservation at grid interfaces on overall solution accuracy were not understood, and it was uncertain that the solution was always the correct solution. However, nonconservative interface treatments have been widely employed with satisfactory results (e.g., Steger, Benek, and Dougherly [12] and Chesshire and Henshaw [13]), even in simulations of flows around complete aircrafts (e.g., Flores and Chaderjian [14]). Moreover, this kind of matching was also proposed for material interfaces (e.g., Karni [15]). Nonconservative treatments such as interpolations are often stable and can usually be implemented in multidimensional Chimera grids (e.g., Steger, Benek, and Dougherly [12]). Therefore, one may ask such questions as how large the conservation error due to a nonconservative interface algorithm may be, whether a convergent numerical approximation obtained with it is a weak solution to the PDE, and if such a treatment can provide correct jumps and locations of discontinuities. These questions concern the most important aspects of a numerical approximation in practice.

The objective of this paper is to answer the above questions. We consider calculations of systems of conservation laws using conservative schemes in subdomains and nonconservative treatments at grid interfaces. In particular, we will examine conservation error of a numerical solution and investigate whether its limit state can be a weak solution, whose conservation error is zero. In calculations of Euler equations, the error is the imbalance of mass, momentum, and energy. When a grid interface is not treated conservatively, conservation error is an important indicator for quality of numerical solutions and can be detected numerically, and it is used by many authors (e.g., Wang [5], Karni [15]).

The rest of the paper is organized as follows. Section 2 deals with one-dimensional cases. In this section, two nonconservative interface algorithms are discussed, one
being an interpolation and the other being a perturbation of a conservative interface matching. It is shown that the conservation error of a numerical solution due to nonconservation at a grid interface has an upper bound if the solution itself is bounded. By the way of Lax and Wendroff [16], it is proven that the limit of the solution can be a weak solution to the PDE under one of three conditions, two of which are a requirement on the state of the numerical solution near the interface and the other one is a restriction on the interface algorithm. These conditions do not tell exactly how to construct a nonconservative interface algorithm to ensure that a limit is a weak solution, but they present a way to detect numerically whether the limit, if exists, will be a weak solution. In section 3, these results are extended to two-dimensional cases in which two grids intersect in arbitrary ways. Then, to illustrate our analysis, numerical examples are displayed in section 4. The conclusions of the paper are straightforward and we believe that they shed light on practical calculations.

2. One-dimensional case. Consider the initial value problem for a one-dimensional system of conservation laws

\[ U_t(x, t) + F_x(U(x, t)) = 0, \quad t, x \in (0, +\infty) \times (-\infty, +\infty), \]
\[ U(x, 0) = U_0(x), \]

where \( U_0(x) \) tends fast to a constant when \( x \) goes to infinity and \( F(U(x, t)) \) is bounded provided \( U(x, t) \) is bounded. Also, in this paper we always assume that a system of conservation laws only has piecewise smooth solution. In Figure 2.1, the x-axis is partitioned into subdomains \( A \) and \( B \) that patch with each other at zonal interface \( x_I \), where either the grid or the numerical scheme or both switch. We use explicit schemes in conservation form in the two subdomains and let \( Q \) be the zonal interface condition. Hence,

\[ V_i^{n+1} = \begin{cases} 
V_i^n - \lambda_A(E_{i+1/2}^A - E_{i-1/2}^A), & i \leq I - 1, \\
V_i^n - \lambda_B(E_{i+1/2}^B - E_{i-1/2}^B), & i \geq I + 1, \\
Q_i^{n+1}, & i = I,
\end{cases} \]

where

\[ V_i^0 = \begin{cases} 
\frac{1}{\Delta x_A} \int_{x_i-1/2}^{x_i+1/2} U_0(x)dx, & i \leq I - 1, \\
\frac{1}{\Delta x_B} \int_{x_i-1/2}^{x_i+1/2} U_0(x)dx, & i \geq I + 1, \\
\frac{2}{\Delta x_A + \Delta x_B} \int_{x_i-1/2}^{x_i+1/2} U_0(x)dx, & i = I,
\end{cases} \]

where super- and subscript \( A \) and \( B \) stand for subdomains \( A \) and \( B \), respectively, superscript \( n \) and subscript \( i \) refer to time \( t^n \) and grid node \( x_i \), respectively, \( \lambda = \Delta t/\Delta x \), \( \Delta t \) is the time step, \( \Delta x \) is the grid spacing, and \( x_{i+1/2} = (x_i + x_{i+1})/2 \). In (2.2a), \( E_{i+1/2}^A \) and \( E_{i+1/2}^B \) are differentiable with respect to each of their arguments. Consistency requires that

\[ E_{i+1/2}^A(a, \ldots, a) = F(a), \\
E_{i+1/2}^B(a, \ldots, a) = F(a). \]
In (2.2a), the interior schemes have a stencil of $2k + 1$. When $k > 1$, in order to calculate $E_{l-1/2}^A$ and $E_{l+1/2}^B$, the following linear interpolation is used:

\[ V_I' = \frac{x_I' - x_i}{\Delta x_B} (V_{i+1} - V_i) + V_i, \quad x_i < x_I' \leq x_{i+1}, i \geq I, \]

\[ V_I'' = \frac{x_I'' - x_{i-1}}{\Delta x_A} (V_i - V_{i-1}) + V_{i-1}, \quad x_{i-1} < x_I'' \leq x_i, i \leq I. \]

The methods to determine the above $Q_i^{n+1}$ fall into two categories. The first one is a conservative treatment and it always guarantees a so-called conservation property. The following is the typical conservative algorithm proposed in [1, 2, 3]:

\[ Q_i^{n+1} = V_i^n - \frac{2\Delta t}{\Delta x_A + \Delta x_B} (E_{l+1/2}^B - E_{l-1/2}^A). \]

The second one is a nonconservative matching, which in general fails to ensure the property. Interpolation is such a matching often adopted. Here we consider

\[ Q_i^{n+1} = \frac{\Delta x_A}{\Delta x_A + \Delta x_B} (V_{l+1}^{n+1} - V_{l-1}^{n+1}) + V_{l-1}^{n+1}, \]

which is a linear interpolation. Compared with conservative methods, interpolation is easier and always possible to implement, especially in calculations with complex geometries, and it usually gives stable and less oscillatory solutions. Actually, we have the following results.

**Theorem 2.1.** Suppose $U(x, t)$ in (2.1) is a scalar function.

1. Let $TV^c(V_i^{n+1})$ be the total variation of $V_i^{n+1}$ obtained with (2.3) and $TV^{nc}(V_i^{n+1})$ be that of $V_i^{n+1}$ obtained with (2.4); then, for a given $V_i^n$,

\[ TV^{nc}(V_i^{n+1}) \leq TV^c(V_i^{n+1}). \]

2. Let (2.2a) consist of monotone schemes for both subdomains $A$ and $B$ and (2.4) for their interface. Then, if $V_i^n$ is strictly monotone, $V_i^{n+1}$ will be strictly monotone under the restriction

\[ \left| \frac{\lambda_A(E_{l-1/2}^A - E_{l-3/2}^A) - \lambda_B(E_{l+3/2}^B - E_{l+1/2}^B)}{V_{l+1}^n - V_{l-1}^n} \right| < 1. \]
Proof. (1). In view of (2.4), one has
\[
TV^{nc}(V_i^{n+1}) = \sum_{i \leq I-1,i \geq I+2} |V_i^{n+1} - V_{i-1}^{n+1}| + |V_{i+1}^{n+1} - V_i^{n+1}|
\]
\[
\leq \sum_{i \leq I-1,i \geq I+2} |V_i^{n+1} - V_{i-1}^{n+1}|
\]
\[
+ |V_{i+1}^{n+1} - V_i^{n+1}| + |V_{I-1}^{n+1} - V_{I-1}^{n+1}|
\]
\[
= TV^{nc}(V_i^{n+1}).
\]

(2) If \( V_i^n \) is strictly monotone increasing (decreasing), then it is seen from (2.2d) that \( V_i^n \) and \( V_i^n \) are also strictly monotone increasing (decreasing). The interior schemes can be written as
\[
V_i^{n+1} = P^A(V_{i-k}, \ldots, V_{i+k}), \quad i \leq I - 1,
\]
\[
V_i^{n+1} = P^B(V_{i-k}, \ldots, V_{i+k}), \quad i \geq I + 1.
\]
Due to the mean value theorem, there exists a number \( 0 < \alpha < 1 \) such that
\[
V_i^{n+1} - V_i^{n+1} = \sum_{k=1}^{2k+1} P_k^A(aV_{i-k} + (1 - \alpha)V_{i-1-k} + \ldots, aV_{i+k} + (1 - \alpha)V_{i+k-1})
\]
\[
(V_{i+k-1} - V_{i+k-2}), \quad i \leq I - 1,
\]
where \( P_k^A \) is the derivative of \( P^A \) over its \( k \)th argument. By the definition of monotonicity, it is known that \( V_i^{n+1} \) is strictly monotone increasing (decreasing). Similarly, it is known that \( V_i^{n+1} \) is strictly monotone increasing (decreasing), too. Using (2.6), one has
\[
V_i^{n+1} - V_i^{n+1} < 0, \quad \text{when} \quad V_i^{n+1} - V_i^{n+1} < 0,
\]
\[
V_i^{n+1} - V_i^{n+1} > 0, \quad \text{when} \quad V_i^{n+1} - V_i^{n+1} > 0.
\]
Therefore, in view of (2.4), \( V_i^{n+1} \) remains strictly monotone. This completes the proof of the theorem. \( \square \)

In general, an explicit nonconservative interface treatment can be written as
\[
Q_i^{n+1} = V_i^n - \frac{2\Delta t}{\Delta x_A + \Delta x_B} (E_{i+1/2}^B - E_{i-1/2}^A) + q_i^n,
\]
where \( \Delta_1 = \max(\Delta x_A, \Delta x_B, \Delta t) \). It is seen that a nonconservation matching may be considered as a modification of conservative scheme (2.3). In section 4 we will give one of its concrete forms.

By using \( V_i^n \) we define grid functions
\[
U_\Delta(x,t) = V_i^n, \quad x,t \in [x_{i-1/2}, x_{i+1/2}] \times [t^n, t^{n+1}]
\]
and
\[
F_\Delta(x,t) = \begin{cases} 
E_{i+1/2}^A, & x,t \in [x_i, x_{i+1}) \times [t^n, t^{n+1}], \quad i \leq I - 1, \\
E_{i+1/2}^B, & x,t \in [x_i, x_{i+1}) \times [t^n, t^{n+1}], \quad i \geq I.
\end{cases}
\]

**Lemma 2.1.** Let \( U_\Delta(x,t) \) be bounded; then \( F_\Delta(x,t) \) is also bounded. Moreover, let \( U_\Delta(x,t) \) converge to a function \( U(x,t) \) as \( \Delta_1 \to 0 \); then, at a point \( x \) where \( U(x,t) \) is continuous,
\[
\lim_{\Delta_1 \to 0} F_\Delta(x,t) = F(U(x,t)),
\]
and at a point where it is discontinuous,

\begin{equation}
\lim_{\Delta_{1-0}} F^\Delta(x, t) = F(U(x, t)) + O(U(x^1, t) - U(x^-, t)).
\end{equation}

\textbf{Proof.} It follows from (2.13), the mean-value theorem, and (2.2c) that

\begin{equation}
F^\Delta(x, t) = E(U^\Delta(x - (k - 1/2)\Delta x, t), \ldots, U^\Delta(x + (k - 1/2)\Delta x, t)) = E(U(x, t), \ldots, U(x, t))
+ \sum_{k' = 1}^{2k} E_{k'} \times (U^\Delta(x + (k' - 1/2)\Delta x, t) - U^\Delta(x, t))
+ \sum_{k' = 1}^{2k} E_{k'} \times (U^\Delta(x + (k' - 1/2)\Delta x, t) - U^\Delta(x, t)).
\end{equation}

In view of (2.15), the proof is obvious. \( \Box \)

Using (2.12) and (2.13), we can write (2.2a) as

\begin{equation}
U^\Delta(x, t + \Delta t) = \begin{cases}
U^\Delta(x, t) - \lambda_A(F(x + \frac{\Delta x_A}{2}, t) - \frac{\lambda_A}{2} F(x - \frac{\Delta x_A}{2}, t)), & x \in (-\infty, x_{I-1/2}), \\
U^\Delta(x, t) - \lambda_B(F(x + \frac{\Delta x_B}{2}, t) - \frac{\lambda_B}{2} F(x - \frac{\Delta x_B}{2}, t)), & x \in [x_{I+1/2}, +\infty), \\
Q^+_{I+1}, & x \in [x_{I-1/2}, x_{I+1/2}].
\end{cases}
\end{equation}

Integrating conservation laws (2.1) and using the Green formula give rise to

\begin{equation}
\int_{-\infty}^{+\infty} (U(x, t_2) - U(x, t_1)) dx + C_1 = 0,
\end{equation}

where \( C_1 = (t_2 - t_1)(F(U_0(+\infty)) - F(U_0(-\infty))) \) and \( t_2 \geq t_1 \geq 0 \). Equation (2.17) indicates that the solution of (2.1) has a so-called conservation property that is stated in the following proposition.

\textbf{Proposition 2.1.} Let \( U(x, t) \) be a solution of the initial value problem (2.1); then (2.17) holds.

If \( U_0(-\infty) = U_0(+\infty) \), it is known from (2.17) that the integration of the solution to (2.1) over the whole x-axis remains unchanged. In view of (2.17), we define the conservation error \( U^\Delta(x, t) \) as

\begin{equation}
S_{1\Delta}(t_1, t_2) = \int_{-\infty}^{+\infty} (U^\Delta(x, t_2) - U^\Delta(x, t_1)) dx + C_1.
\end{equation}

If \( Q^+_{I+1} \) in (2.2a) is chosen such that

\begin{equation}
S_{1\Delta}(t_1, t_2) = 0,
\end{equation}

it is called a conservative interface algorithm. Otherwise, it is referred to as a non-conservative interface algorithm. It can be verified that (2.3) guarantees (2.19), while in general (2.4) and (2.11) cannot. When \( Q^+_{I+1} \) is chosen as (2.4), the error between two time steps reads

\begin{equation}
S_{1\Delta}(t^n, t^{n+1}) = \frac{\Delta t \Delta x_B}{\Delta x_A} (E^A_{I-3/2} - E^A_{I-1/2}) + \Delta t (E^A_{I-1/2} - E^B_{I+1/2})
+ \frac{\Delta t \Delta x_A}{\Delta x_B} (E^B_{I+1/2} - E^B_{I+3/2}),
\end{equation}
and its magnitude depends on numerical solution state near the interface. When there is no discontinuity near \( x_I \) and \( V_i - V_{i+1} = O(\Delta_1) \), it is known from (2.2c) that
\[
(2.21) \quad S_1(\Delta(t^n, t^{n+1}) = O(\Delta_1^2).
\]

When there is a discontinuity near \( x_I \), (2.21) is usually no longer true and \( S_1(\Delta(t^n, t^{n+1}) \) becomes larger in magnitude (see [8, 17]). A numerical example will be given in Figure 4.1 in section 4.

For a calculation of the initial value problem (2.1), conservation error of the numerical solution at time \( T \) is \( S_1(\Delta(0, T)) \). It follows from (2.2b) and Proposition 2.1 that
\[
(2.22) \quad S_1(\Delta(0, T)) = \int_{x_I-\Delta/2}^{x_I+\Delta/2}(U(x,T) - U(x,0))\,dx.
\]

This suggests that \( S_1(\Delta(0, T)) \) is a kind of numerical error. In case that \( U(x,t) \) in (2.1) is a scalar function, (2.22) yields
\[
(2.23) \quad |S_1(\Delta(0, T))| \leq \|U(x,T) - U(x,0)\|_1.
\]

Here and hereinafter, \( \| \cdot \|_1 \) stands for \( L_1 \)-norm of a scalar function. Equation (2.23) indicates that a large conservation error means a large numerical error, which may appear as an obvious inaccuracy for shock jumps and locations. In the following, we will discuss in general how large the conservation error may be if a nonconservative interface algorithm is used.

**Theorem 2.2.** If \( U_\Delta(x,t) \) is uniformly bounded within a neighborhood of \( x_I \), then \( S_1(\Delta(0, T)) \) is bounded, too.

**Proof.** Using (2.16) it is known that
\[
(2.24) \quad S_1(\Delta(0, T)) = \sum_{n=0}^{N-1} S_1(n\Delta t, (n+1)\Delta t)
\]

\[
= \int_0^T \left( F_\Delta(x_{I-1/2}, t) - F_\Delta(x_{I+1/2}, t) \right) dt + \int_{x_{I-1/2}}^{x_{I+1/2}} (U_\Delta(x,T) - U_\Delta(x,0))\,dx,
\]

where \( \Delta t = T/N \), \( N \) being the total number of time steps. Equation (2.24) yields
\[
(2.25) \quad \|S_1(\Delta(0, T))\|_\infty \leq \int_0^T \|F_\Delta(x_{I-1/2}, t) - F_\Delta(x_{I+1/2}, t)\|_\infty dt + \int_{x_{I-1/2}}^{x_{I+1/2}} \|U_\Delta(x,T) - U_\Delta(x,0)\|_\infty dx.
\]

Here and hereinafter, \( \| \cdot \|_\infty \) refers to maximum norm of a vector. In view of the given condition and Lemma 2.1, let
\[
(2.26) \quad \|U_\Delta(x,t)\|_\infty < m_1, \quad \|F_\Delta(x,t)\|_\infty < m_2,
\]

where \( m_1 \) and \( m_2 \) are constants independent of \( x \) and \( t \). Then, it follows from (2.25) that
\[
(2.27) \quad \|S_1(\Delta(0, T))\|_\infty \leq M,
\]

in which \( M = m_1(\Delta x_A + \Delta x_B) + 2m_2T \). This completes the proof of the theorem. \( \square \)
Theorem 2.2 implies that, if \( U_\Delta(x,t) \) remains bounded at a neighborhood of \( x_I \), conservation error \( S_{1\Delta}(0,T) \) will not accumulate to an infinite amount as mesh size tends to zero. Instead, we have the following result.

**Theorem 2.3.** Suppose that \( U_\Delta(x,t) \) is uniformly bounded and it converges to a function \( U(x,t) \) as the mesh size approaches zero. Then, either of the following conditions is sufficient for \( U(x,t) \) to be a weak solution of the initial value problem (2.1):

A. \( U(x_I,t) \) is continuous, except at the moment \( t_m \), where \( m \) is at most countable.

B. \( U(x_I,t) \) is always discontinuous but

\[
\lim_{\Delta \to 0} (F_\Delta(x_{I-1/2},t) - F_\Delta(x_{I+1/2},t)) = 0 \quad \text{almost everywhere (a.e.).}
\]

**Proof.** Multiplying the first two branches of (2.16) with a scalar function \( \Phi(x,t) \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+) \), integrating the resulting equality over \( (-\infty, +\infty) \times (0, +\infty) \), and introducing the changes of variables yield

\[
\begin{align*}
\int_{\Delta t}^{-\Delta t} \int_{x_{I-1}}^{x_{I+1}} \left( &\frac{\Phi(x,t - \Delta t) - \Phi(x,t)}{\Delta t} U_\Delta(x,t) \\
&+ \frac{\Phi(x - \Delta x/2, t) - \Phi(x + \Delta x/2, t)}{\Delta x} F_\Delta(x,t) \right) dx dt \\
+ &\int_{\Delta t}^{\Delta t} \int_{x_{I-1}}^{x_{I+1}} \left( &\frac{\Phi(x,t - \Delta t) - \Phi(x,t)}{\Delta t} U_\Delta(x,t) \\
&+ \frac{\Phi(x - \Delta x/2, t) - \Phi(x + \Delta x/2, t)}{\Delta x} F_\Delta(x,t) \right) dx dt \\
- &\int_0^{\Delta t} \int_{-\infty}^{x_{I-1}} \Phi(x,t) U_\Delta(x,t) dx dt \\
- &\int_0^{\Delta t} \int_{x_{I+1}}^{\infty} \Phi(x,t) U_\Delta(x,t) dx dt \\
+ &\int_0^{\Delta t} (\Phi(\alpha + \Delta x/2, t) F_\Delta(x_{I-1/2}, t) - \Phi(\beta - \Delta x/2, t) F_\Delta(x_{I+1/2}, t)) dt = 0,
\end{align*}
\]

where \( \alpha \in (x_{I-1}, x_I) \) and \( \beta \in (x_I, x_{I+1}) \). Let \( \Delta t \to 0 \), then, in view of the Lebesgue bounded convergence theorem, Lemma 2.1, and (2.2b), (2.29) becomes

\[
\begin{align*}
\int_0^{\Delta t} \int_{-\infty}^{x_{I-1}} (\Phi_1(x,t) U(x,t) + \Phi_2(x,t) F(U(x,t))) dx dt + &\int_0^{\Delta t} \Phi(x,0) U_0(x) dx \\
+ &\int_0^{\Delta t} \Phi(x,t) \lim_{\Delta \to 0} (F_\Delta(x_{I-1/2}, t) - F_\Delta(x_{I+1/2}, t)) dt = 0.
\end{align*}
\]

Since \( \Phi(x,t) \) is any function belonging to \( C_0^\infty(\mathbb{R} \times \mathbb{R}^+) \), (2.30) suggests that \( U(x,t) \) is a weak solution of (2.1) if and only if

\[
\lim_{\Delta t \to 0} (F_\Delta(x_{I-1/2}, t) - F_\Delta(x_{I+1/2}, t)) = 0 \quad \text{a.e.}
\]

Because either condition A. or B. leads to (2.31), \( U(x,t) \) is a weak solution of (2.1). This completes the proof of Theorem 2.3.

From Proposition 2.1 or (2.23), it is known that conservation error of a piecewise smooth function \( U(x,t) \) is zero if it is a weak solution to (2.1). Therefore, under condition A. or B., conservation error \( S_{1\Delta}(0,T) \) of a numerical approximation obtained with a nonconservative interface treatment decreases with the mesh size. As a result, although the approximation does not satisfy (2.19), it is not wrong but only
has numerical errors. In this case, as to be shown in section 4, inaccuracy for shock locations and jumps can be reduced by mesh refinement.

Conditions $A_1$ and $B_1$ in Theorem 2.3 correspond to two typical situations in practical calculations; in the former shocks and contact discontinuities propagate across a grid interface (for at most countable times), while in the latter a stationary shock or a contact discontinuity stops right at an interface. It is easy to verify that condition $B_1$ contains the following as its special case:

$$U(x, t)$$ has a stationary discontinuity at $x_I$, but

$$\lim_{\Delta_1 \to 0} F_\Delta(x_{I \pm 1/2}, t) = F(U(x_I^+, t)),$$

and Rankine–Hugoniot conditions are satisfied.

A situation about the limit state of $U_\Delta(x, t)$ that does not fall into the categories of $A_1$ and $B_1$ is when a stationary discontinuity is caught right at the interface but (2.28) is violated. At this time, the numerical solution may converge, as that stated by Hou and Le Floch [10], to the solution of another system of inhomogeneous conservation laws. This situation arises from two sources. In the first source the physical problem has a discontinuity propagating through the interface, but, due to a nonconservative matching, the computed discontinuity cannot pass across and it stops there. In the second source, the physical solution itself possesses a stationary discontinuity at the interface, but the nonconservation leads to a numerical discontinuity there with a wrong jump. We will present a numerical example for the second source in section 4.

In order to avoid the situation, it is better to devise a grid interface in such a way that it has a distance from discontinuities if their positions can be estimated before computations.

With the aid of Theorem 2.3 we can check whether the limit of a numerical solution will be a weak solution. Actually, we may do mesh refinement tests to detect that if $A_1$ or $B_1$ is satisfied. In the detection (2.31) is a useful tool, since we know from the proof of the theorem that both conditions $A_1$ and $B_1$ lead to it. In general, it is difficult to show how to make conditions $A_1$ and $B_1$ be satisfied. Some authors have established numerical evidence that conditions can be satisfied (e.g., Karni [15]). In the following we show two simple examples in which interpolation (2.4) is adopted and the two conditions are satisfied.

**Example 1.** Consider the initial value problem of a scalar equation:

$$U_t(x, t) + U_x(x, t) = 0, \quad t, x \in (0, +\infty) \times (-\infty, +\infty),$$

$$U(x, 0) = \begin{cases} -1, & x < 0, \\ 1, & x > 0. \end{cases}$$

(2.33)

Let $x_1 = 0, x_I = 1,$ and $\Delta x_A = \Delta x_B$. Adopting the Lax–Friedrichs scheme and letting Courant number $CFL$ equal 1 in the two subdomains, one has

$$V^{n+1}_i = V^n_{i-1}, \quad i \neq I.$$

(2.34a)

Employing the matching

$$V^{n+1}_I = \frac{V^{n+1}_{I-1} + V^{n+1}_{I+1}}{2},$$

(2.34b)

which is actually (2.4), then one can start to calculate this problem. It is easy to check that as mesh size goes to zero, $A_1$ will be satisfied and, consequently, the limit of the numerical solution is a weak solution to (2.33).
Example 2. Consider the initial value problem of Burger’s equation:

\[
    U_t(x, t) + \left( \frac{1}{2} U(x, t)^2 \right)_x = 0, \quad t, x \in (0, +\infty) \times (-\infty, +\infty),
\]

(2.35)

\[
    U(x, 0) = \\
    \begin{cases}
    1, & x < 0, \\
    -1, & x > 0.
    \end{cases}
\]

The solution of (2.35) has a stationary discontinuity at \( x = 0 \). Let \( x_I = 0 \), and, again, set \( \Delta x_A = \Delta x_B \) and adopt the Lax–Friedrichs scheme in each subdomain. Let \( V_I^0 = 0 \) and use (2.4), then some straightforward calculations yield

\[
    V_I^{n+1} = 0, \quad V_{I-1}^{n+1} = -V_{I+1}^{n+1},
\]

which gives rise to

\[
    F_\Delta(x_{I-1/2}, t) = F_\Delta(x_{I+1/2}, t);
\]

i.e., condition \( B_1 \) is satisfied. It is also obvious that the numerical solution will converge to the solution of (2.35). Actually, (2.19) is satisfied in this example.

Theorem 2.3 provides a way to judge whether a convergent solution will be a weak solution via conditions \( A_1 \) and \( B_1 \), which are requirements on numerical solution states near interfaces. But, can we make the judgment according to an interface algorithm itself? Actually, we have the following theorem.

**Theorem 2.4.** Let \( Q_I^{n+1} \) be (2.11) and \( U_\Delta(x, t) \) be uniformly bounded and converge to a function \( U(x, t) \) as the mesh size approaches zero. Then, \( U(x, t) \) is a weak solution if and only if the following condition holds:

\[
    \lim_{\Delta_1 \to 0} q_I^n = 0
\]

for almost all \( n \).

**Proof.** Let \( q(t) = q_I^n, t^n < t \leq t^{n+1} \). Using (2.30), (2.11), we obtain that

\[
    \int_0^{+\infty} \Phi(x_I, t) \lim_{\Delta_1 \to 0} q(t) dt = 0,
\]

which makes the proof to be obvious. \( \square \)

Theorem 2.4 states that a nonconservative interface algorithm can provide a weak solution only if it satisfies condition (2.38), which can be detected numerically. Condition (2.38) suggests a rule for construction of the algorithm; it should be “almost” a conservative treatment, or a “small” perturbation of a conservative matching, and tend to be conservative when the mesh shrinks. As to be shown in section 4, a scheme resulting from the perturbation may give better accuracy and less oscillations than the conservative matching. The so-called adaptive interface scheme can also be considered as such an algorithm; it is conservative when there is a discontinuity near an interface; otherwise, it is not (e.g., Karni [15], Tang and Lee [7]). Moreover, the theorem implies the statement, proposed by Berger [3], that the limit of a numerical solution obtained with interface matching (2.3) is a weak solution to (2.1).

3. Two-dimensional case. In this section, the above results about nonconservative interface algorithms will be extended to two-dimensional cases. Consider now the initial value problem for a two-dimensional system of conservation laws:

\[
    U_t(x, y, t) + G_x(U(x, y, t)) + H_y(U(x, y, t)) = 0, \quad 0 < t < +\infty, \quad x, y \in \Omega,
\]

\[
    U(x, y, 0) = U_0(x, y).
\]

(3.1)
Here $U_0(x, y)$ has a bounded support, and $G$ and $H$ are differentiable with respect to $U(x, y, t)$. Let $\Omega$ be separated by a smooth curve $\Gamma$, a zonal interface, into $\Omega_A$ and $\Omega_B$ (Figure 3.1). In $\Omega_A$ the governing equation is written as (3.1), while in subdomain $\Omega_B$ it is expressed in another coordinate system:

$$(3.2a) \quad \overline{U}_t(\xi, \eta, t) + \overline{G}(\overline{U}(\xi, \eta, t)) + \overline{H}(\overline{U}(\xi, \eta, t)) = 0,$$

where

$$(3.2b) \quad \overline{U}(\xi, \eta, t) = JU(x, y, t),$$

$$(3.2c) \quad \overline{G}(\overline{U}(\xi, \eta, t)) = J(\xi_x G(U(x, y, t)) + \xi_y H(U(x, y, t))),$$

$$(3.2d) \quad \overline{H}(\overline{U}(\xi, \eta, t)) = J(\eta_x G(U(x, y, t)) + \eta_y H(U(x, y, t))).$$

and

$$(3.3) \quad J = x\xi y\eta - x\eta y\xi,$$

$$(3.4) \quad \xi_x J = y\eta, \quad \xi_y J = -x\eta, \quad \eta_x J = -y\xi, \quad \eta_y J = x\xi.$$

As depicted in Figure 3.1, the grid for $\Omega_A$ is $G_A$, composed by lines $x = x_i$ and $y = y_j$, and the grid for $\Omega_B$ is $G_B$, made up of lines $\xi = \xi_i$ and $\eta = \eta_j$. Here $x_i$, $y_j$, $\xi_i$, and $\eta_j$ are constants. In the figure, grids $G_A$ and $G_B$ intersect with each other in an arbitrary way, $\Gamma$ intersects with each grid line at most once, and it is nearly straight between two adjacent grid lines. In Figure 3.1, for each $i$, grid $G_A$ has a node $(i, j_A(i))$ located in $\Omega_B$; thus its node $(i, j_A(i) + 1)$ is located in $\Omega_A$, and for each $j$, the grid has a node $(i_A(j), j)$ located in $\Omega_B$; thus its node $(i_A(j) - 1, j)$ is located in $\Omega_A$. The nodes $(i, j_A(i))$ and $(i_A(j), j)$ are referred to as interface nodes of grid $G_A$ and marked by circles in the figure. Also in this figure, the lines

$$(3.3) \begin{align*}
&\langle x, y \mid x = x_{i_1(j_1)} - 1/2, y_{j_1} - 1/2 \leq y < y_{j_1+1}/2 \rangle, \\
&\langle x, y \mid x_{i_1} - 1/2 \leq x < x_{i_1+1}/2, y = y_{j_1(i_1+1)/2} \rangle
\end{align*}$$
comprise a step line $$\Gamma_A$$. Let grid line $$x_i$$ intersects with $$\Gamma$$ and $$\Gamma_A$$ at points $$\Gamma_i$$ and $$\Gamma_{AI}$$, respectively, and grid line $$y_j$$ intersects with $$\Gamma$$ and $$\Gamma_A$$ at points $$\Gamma_j$$ and $$\Gamma_{AJ}$$, respectively. It is readily seen that

$$\forall i,j \in \{i_A, j_A\} \setminus \{i_{AI}, j_{AI}\},$$

$$|y^{\Gamma_{AI}} - y^{\Gamma_i}| < \Delta y/2, \quad |x^{\Gamma_{AJ}} - x^{\Gamma_j}| < \Delta x/2.$$  

Besides, let $$\Gamma_B$$ be $$\xi = \xi^B + \Delta \xi/2$$, and interface nodes of grid $$G_B$$ are those on $$\Gamma_B$$. In the following, $$\{i_A, j_A\}$$ and $$\{i_B, j_B\}$$ stand for all nodes of grids $$G_A$$ and $$G_B$$, respectively, $$\{i_{AI}, j_{AI}\}$$ and $$\{i_{BJ}, j_{BJ}\}$$ represent nodes of grid $$G_A$$ that are located in $$\Omega_B$$ and nodes of grid $$G_B$$ that are located in $$\Omega_A$$, respectively.

In subdomains $$A$$ and $$B$$, explicit conservative difference schemes with a stencil of $$2k + 1$$ are used

$$V^{n+1}_{i,j} = \frac{1}{\Delta x} \int_{x_i-\Delta x/2}^{x_i+\Delta x/2} \int_{y_j-\Delta y/2}^{y_j+\Delta y/2} U_0(x, y) dxdy,$$

where

$$V^0_{i,j} = \frac{1}{\Delta x \Delta y} \int_{x_i-\Delta x/2}^{x_i+\Delta x/2} \int_{y_j-\Delta y/2}^{y_j+\Delta y/2} U_0(x, y) dxdy,$$

Here $$\lambda_x = \Delta t/\Delta x$$, $$\lambda_y = \Delta t/\Delta y$$, $$\lambda_\xi = \Delta t/\Delta \xi$$, $$\lambda_\eta = \Delta t/\Delta \eta$$, and $$U_0(\xi, \eta) = JU_0(x, y)$$. To calculate (3.5a) at nodes near interface $$\Gamma$$ (e.g., those marked with crosses in Figure 3.1), explicit interface schemes that are not necessarily conservative are employed

$$V^{n+1}_{i,j} = Q^{n+1}_{i,j}, \quad i,j \in \{i_{AI}, j_{AI}\},$$

$$\nabla^{n+1}_{i,j} = \nabla^{n+1}_{i,j}, \quad i,j \in \{i_{BJ}, j_{BJ}\}.$$  

Assume that $$Q^{n+1}_{i,j}$$ and $$Q^{n+1}_{i,j}$$ have a finite stencil and they are bounded if each of their argument is bounded. This is true when $$Q^{n+1}_{i,j}$$ and $$Q^{n+1}_{i,j}$$ are given by interpolation.

Using grid functions, we write (3.5a) as

$$\frac{U^A_{i,j}(x, y, t + \Delta t)}{U^A_{i,j}(x, y, t)} - \lambda_x(G^A_{i,j}(x + \Delta x/2, y, t) - G^A_{i,j}(x - \Delta x/2, y, t)) - \lambda_y(H^A_{i,j}(x, y + \Delta y/2, t) - H^A_{i,j}(x, y - \Delta y/2, t)),$$

where

$$\Omega_{AD} = \{x, y \mid x \leq x^{\Gamma_A}, y \geq y^{\Gamma_A}\},$$

$$\Omega_{BD} = \{\xi, \eta \mid \xi \geq \xi^{\Gamma_B}\},$$

$$\xi^{\Gamma_B} = \xi^B + \Delta \xi/2.$$
and $\Omega_{AD}$ is the area at the left side of $\Gamma_A$, $\Omega_{BD}$ is the area at the right side of $\Gamma_B$. Furthermore, we define that

$$U_\Delta(x, y, t) = \begin{cases} 
U^A_\Delta(x, y, t), & x, y \in \Omega_A, \\
U^B_\Delta(\xi, \eta, t), & \xi, \eta \in \Omega_B.
\end{cases}$$

Details of (3.7), (3.8), and (3.9) are given in [17]. Following the same procedure in the one-dimensional case, we obtain the following lemma.

**Lemma 3.1.** Let

$$\|U_\Delta(x, y, t)\|_\infty < m_1, \quad \|U_\Delta(\xi, \eta, t)\|_\infty < m_2;$$

then

$$\|G^A_\Delta(x, y, t)\|_\infty < m_3, \quad \|H^A_\Delta(x, y, t)\|_\infty < m_4,$$

$$\|G^B_\Delta(\xi, \eta, t)\|_\infty < m_5, \quad \|H^B_\Delta(\xi, \eta, t)\|_\infty < m_6,$$

where $m_1, \ldots, m_6$ are constants independent of $x, y, \xi, \eta,$ and $t$. Moreover, let $U_\Delta(x, y, t)$ or $U_\Delta(\xi, \eta, t)$ converge to a function $U(x, y, t)$ or $\overline{U}(\xi, \eta, t)$. Then, at a point $(x, y)$ where $U(x, y, t)$ or $\overline{U}(\xi, \eta, t)$ is continuous,

$$\lim_{\Delta \to 0} G^A_\Delta(x, y, t) = G(U(x, y, t)),$$

$$\lim_{\Delta \to 0} H^A_\Delta(x, y, t) = H(U(x, y, t)),$$

$$\lim_{\Delta \to 0} G^B_\Delta(\xi, \eta, t) = \overline{C}(U(\xi, \eta, t)),$$

$$\lim_{\Delta \to 0} H^B_\Delta(\xi, \eta, t) = \overline{C}(\overline{U}(\xi, \eta, t))$$

and at a point $(x, y)$ where it is discontinuous,

$$\lim_{\Delta \to 0} G^A_\Delta(x, y, t) = G(U(x, y, t)) + O(U(x^+, y, t) - U(x^-, y, t)),$$

$$\lim_{\Delta \to 0} H^A_\Delta(x, y, t) = H(U(x, y, t)) + O(U(x^+, y, t) - U(x^-, y, t)),$$

$$\lim_{\Delta \to 0} G^B_\Delta(\xi, \eta, t) = \overline{C}(U(\xi, \eta, t)) + O(\overline{C}(\xi^+, \eta, t) - \overline{C}(\xi^-, \eta, t)),$$

$$\lim_{\Delta \to 0} H^B_\Delta(\xi, \eta, t) = \overline{C}(\overline{U}(\xi, \eta, t)) + O(\overline{C}(\xi^+, \eta, t) - \overline{C}(\xi^-, \eta, t)),$$

where $\Delta_2 = \max(\Delta x, \Delta y, \Delta \xi, \Delta \eta, \Delta t)$.

Similar to the one-dimensional case, we define the conservation error of $U_\Delta(x, y, t)$ between time $t_1$ and $t_2$ as ($t_2 \geq t_1 \geq 0$)

$$S_{2\Delta}(t_1, t_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (U_\Delta(x, y, t_2) - U_\Delta(x, y, t_1)) dxdy.$$

If an interface treatment can guarantee that $S_{2\Delta}(t_1, t_2)$ equals zero, then it is conservative. Otherwise, it is nonconservative. For the case in Figure 3.1, it is derived that

$$S_{2\Delta}(t_1, t_2) = \int_{\Omega_T} (U_\Delta(x, y, t_2) - U_\Delta(x, y, t_1)) dx dy$$

$$+ \int_{\Omega_{AB}} (U_\Delta(\xi, \eta, t_2) - U_\Delta(\xi, \eta, t_1)) d\xi d\eta$$

$$+ \int_{\Omega_{T}} (U_\Delta(x, y, t_2) - U_\Delta(x, y, t_1)) dx dy,$$

where $\Omega_T = \Omega \setminus \Omega_{AD} \setminus \Omega_{BD}$. Since $U_0(x, y)$ has a bounded support and (3.5) and (3.6) are explicit schemes with finite stencils, there must be a bounded region $\Omega_T$ outside which $U_\Delta(x, y, t)(0 \leq t \leq T)$ equals zero. For convenience, let

$$\Gamma_{Tx} = \{ y \mid y \in \Gamma_A \cap \Omega_T \},$$

$$\Gamma_{Ty} = \{ x \mid x \in \Gamma_A \cap \Omega_T \},$$

$$\Gamma_{Tz} = \{ \eta \mid \eta \in \Gamma_B \cap \Omega_T \}.$$
Using (3.7), it is obtained that

\[(3.16a) \quad S_{2\Delta}(0, T) = D(T) + I(T),\]

where

\[
\begin{align*}
D(T) &= \int_{\Gamma_{x}\times(0, +\infty)} G_{\Delta}(x^{\Gamma}, y, t) dydt \\
&\quad - \int_{\Gamma_{y}\times(0, +\infty)} H_{\Delta}(x, y^{\Gamma}, t) dxdt \\
&\quad - \int_{\Gamma_{\xi}\times(0, +\infty)} F_{\Delta}(\xi^{\Gamma}, \eta, t) d\eta dt, \\
I(T) &= \int_{\Omega_{AD}} (U_{\Delta}(x, y, T) - U_{\Delta}(x, y, 0)) dx dy.
\end{align*}
\]

\(D(T)\) is actually the numerical flux accumulation along zonal interface \(\Gamma\).

**Theorem 3.1.** Assume that \(U_{\Delta}(x, y, t)\) is uniformly bounded within a neighborhood of \(\Gamma\); then \(S_{2\Delta}(0, T)\) is bounded.

**Proof.** It follows from (3.16) that

\[(3.17) \quad \|S_{2\Delta}(0, T)\|_{\infty} \leq \|D(T)\|_{\infty} + \|I(T)\|_{\infty}.
\]

By the given condition it is known that each integration in (3.16b) is bounded. Actually, let \(L_{A}\) and \(L_{B}\) be the length of \(\Gamma_{A}\) and \(\Gamma_{B}\) contained in \(\Omega_{T}\), respectively. In view of Lemma 3.1, (3.16b) yields

\[(3.18) \quad \|D(T)\|_{\infty} < m_{3}L_{A} + m_{4}L_{A} + m_{5}L_{B}, \]

\[\|I(T)\|_{\infty} < L_{B}(m_{1}\sqrt{\Delta x^{2} + \Delta y^{2}} + m_{2}\Delta \xi),\]

where \(m_{1}, \ldots, m_{5}\) are independent of mesh size. The proof of the theorem is completed by using (3.17) and (3.18).

**Theorem 3.2.** Suppose that \(U_{\Delta}(x, y, t)\) is uniformly bounded and, as \(\Delta_{2}\) tends to zero, it converges to a function \(U(x, y, t)\). Then, either of the following conditions is sufficient for \(U(x, y, t)\) to be a weak solution of the initial value problem (3.1):

A\(_{2}\) \(U(x^{\Gamma}, y^{\Gamma}, t)\) is continuous, except along \(m\) curves on the surface \((x^{\Gamma}, y^{\Gamma}, t)\), where \(m\) is at most countable.

B\(_{2}\) \(U(x^{\Gamma}, y^{\Gamma}, t)\) is always discontinuous but the following is true:

\[(3.19) \quad \lim_{\Delta_{2}\to0} (J_{x}G_{\Delta}^{A}(x^{\Gamma}, y, t) + J_{y}H_{\Delta}^{A}(x, y^{\Gamma}, t) - G_{\Delta}^{B}(\xi^{\Gamma}, \eta, t)) = 0\]

for each \(\eta\).

**Proof.** Let \(\Phi(x, y, t)\) be a scalar function that belongs to \(C_{0}^{\infty}(R^{2} \times R^{+})\) and be expressed as \(\Phi(\xi, \eta, t)\) in coordinate system \((\xi, \eta, t)\). Suppose that \(\Omega_{T}\) is the support of \(\Phi(x, y, t)\). Multiplying the first and second equalities of (3.7) by \(\Phi(x, y, t)\) and \(\Phi(\xi, \eta, t)\), respectively, integrating the resulting equations over \(\Omega_{AD} \times (0, +\infty)\) and \(\Omega_{BD} \times (0, +\infty)\), respectively, changing variables, and applying the mean value theorems, one then has
From Figure 3.1, it is readily seen that on $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$, the last term in (3.20a) becomes

$$\int_{\Omega_{AD} \times (\Delta t, +\infty)} \left( \frac{\Phi(x, y, t - \Delta t) - \Phi(x, y, t)}{\Delta t} U_\Delta^A(x, y, t) \right) dx dy dt$$

$$+ \int_{\Omega_{BD} \times (\Delta t, +\infty)} \left( \frac{\Phi(x - \frac{\Delta x}{2}, y, t) - \Phi(x + \frac{\Delta x}{2}, y, t)}{\Delta x} G_\Delta^A(x, y, t) \right) \left( \frac{\Phi(x, y - \frac{\Delta y}{2}, t) - \Phi(x, y + \frac{\Delta y}{2}, t)}{\Delta y} H_\Delta^A(x, y, t) \right) dx dy dt$$

$$- \int_{\Omega_{AD} \times (0, \Delta t)} \Phi(x, y, t) U_\Delta^A(x, y, t) dx dt$$

$$+ \int_{\Omega_{BD} \times (0, \Delta t)} \left( \Phi(x, y, t - \Delta t) - \Phi(x, y, t) \right) U_\Delta^B(x, y, t) dx dt$$

$$- \int_{\Omega_{AD} \times (0, \Delta t)} \Phi(x, y, t) U_\Delta^A(x, y, t) dx dt$$

where

$$R = \int_{\Gamma_x \times (0, +\infty)} \Phi(x, y, t) G_\Delta^A(x, y, t) dy dt$$

$$- \int_{\Gamma_y \times (0, +\infty)} \Phi(x, y, t) H_\Delta^A(x, y, t) dx dt$$

$$+ \int_{\Gamma_x \times (0, +\infty)} \Phi(x, y, t) G_\Delta^B(x, y, t) dy dt$$

$$+ O(\Delta_2).$$

Let $U_\Delta(x, y, t)$ tend to $U(x, y, t)$ in the coordinate system $(x, y, z)$, or $\bar{U}(\xi, \eta, t)$ in coordinate system $(\xi, \eta, t)$. In view of (3.5b), (3.2b), and (3.2c), the first four integrals in the left-hand side of (3.20a) becomes

$$- \int_0^{+\infty} \int_{\Omega} \left( \Phi(x, y, t) U(x, y, t) + \Phi_x(x, y, t) G(U(x, y, t)) \right) dx dy dt$$

$$+ \int_{\Gamma_x \times (0, +\infty)} \Phi(x, y, t) U_0(x, y) dx dy,$$

and the last term $R$ reads as

$$\int_{\Gamma_x \times (0, +\infty)} \Phi(\xi, \eta, t) \lim_{\Delta_2 \to 0} \left( \frac{\partial}{\partial \xi} G_\Delta^A(x, y, t) \right)$$

$$+ \int_{\Gamma_y \times (0, +\infty)} \Phi(\xi, \eta, t) \lim_{\Delta_2 \to 0} \left( \frac{\partial}{\partial \eta} H_\Delta^A(x, y, t) \right) dy dt.$$

From Figure 3.1, it is readily seen that on $\Gamma_x$ and $\Gamma_y$ there exist some places where

$$G_\Delta^A(x, y, t) = G_\Delta(x, y, t),$$

$$H_\Delta^A(x, y, t) = H_\Delta(x, y, t),$$

and at the other places

$$G_\Delta^A(x, y, t) = \frac{G_\Delta(x, y, t)}{\Delta x} + O(\Delta x),$$

$$H_\Delta^A(x, y, t) = \frac{H_\Delta(x, y, t)}{\Delta y} + O(\Delta y),$$

$$- U_\Delta(x, y, t) + O(\Delta y),$$

$$- U_\Delta(x, y, t) + O(\Delta y).$$
where \( k' = 1, \ldots, 2k \). From Figure 3.1, it is also readily seen that

\[
G^R_\Delta(\xi^\Gamma, \eta, t) = G^R_\Delta(\xi^\Gamma, \eta, t).
\]

If condition A2 is satisfied, according to (3.4) and Lemma 3.1 and in view that the projection of the \( m \) given curves onto the plane \( \eta - t \) only has zero measure, it is then derived from (3.22) that

\[
\lim_{\Delta \to 0} R = \int_{\Gamma_{T\times(0, +\infty)}} \Pi(\xi^\Gamma, \eta, t)(\overline{G}(\xi^\Gamma, \eta, t)) \overline{G}(\xi^\Gamma, \eta, t)) d\eta dt = 0.
\]

If condition B2 is satisfied, it is also known from (3.19) that (3.24) holds. Substituting (3.21) and (3.24) into (3.20a), it is then known that \( U(x, y, t) \) is a weak solution to (3.1). This completes the proof of Theorem 3.2.

It is seen that the results for the two-dimensional case, including constraints A2 and B2, are similar to those for the one-dimensional case.

4. Numerical examples. Consider the initial value problem for a single conservation law:

\[
U_t(x, t) + \left( \frac{1}{2} U^2(x, t) \right)_x = 0,
\]

\[
U(x, 0) = \begin{cases} 
1, & x < 0, \\
1 - x, & 0 \leq x \leq 1, \\
0, & x > 1. 
\end{cases}
\]

Figure 4.1 is obtained by using interface scheme (2.4) at grid interface \( x_I = 2 \), the Lax–Friedrichs and the Lax–Wendroff schemes in subdomains A and B, respectively, \( \Delta x_A = \Delta x_B \), and CFL = 0.95. The exact solution of the problem is continuous at the interface, except at the moment when a shock passes across it. In the calculation the shock is a little lagged; see Figures 4.1(a) and 4.1(b). As predicted in section 2, conservation error takes place mainly when the shock moves across the interface; see Figures 4.1(c) and 4.1(d). The calculation also shows that as the mesh shrinks, condition A1 holds true, conservation error decreases, and accuracy for shock location improves; see Figures 4.1(b) and 4.1(d). At the same time, the numerical solution will converge to the exact one. An evidence for the convergence is given in Table 4.1. The rate of convergence is approximately of the order 1 in \( L_1 \)-norm and it does not differ much from that if (2.3) is used.

Figure 4.2 shows the performance of conservative interface algorithm (2.3) and its perturbation (2.11) in a calculation of a Riemann problem for Euler equations. In the computation, the Lax–Friedrichs and the Lax–Wendroff schemes are used in subdomains A and B, respectively, \( \Delta x_A = 2 \Delta x_B \), and CFL = 0.95. Here

\[
q^n = (\Delta x_A + \Delta x_B)^2 \left( \frac{V^n_{i+1} - V^n_i}{\Delta x_B} - \frac{V^n_i - V^n_{i-1}}{\Delta x_A} \right).
\]

Equation (4.2) may be considered as an artificial viscous term, and it satisfies (2.38) if the numerical solution is bounded. It is seen in Figure 4.2(a) that algorithm (2.3) induces obvious oscillations at the interface, while its perturbation, a nonconservative matching, presents an approximation with less oscillations there. Also, mesh refinement tests suggest that the latter converges to the exact solution of the problem; see Table 4.2 (superscript \( m \) refers to components of \( U_{\Delta}(x, t) \) and \( U(x, t) \)). The rate of
Table 4.1

<table>
<thead>
<tr>
<th>$\Delta x_A, \Delta x_B$</th>
<th>0.2</th>
<th>0.02</th>
<th>0.002</th>
<th>0.0002</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{1\Delta}(0,T)$</td>
<td>$-4.720 \times 10^{-1}$</td>
<td>$-4.754 \times 10^{-2}$</td>
<td>$-4.644 \times 10^{-3}$</td>
<td>$-8.523 \times 10^{-5}$</td>
</tr>
<tr>
<td>$|U\Delta(x,T) - U(x,T)|_1$</td>
<td>$5.276 \times 10^{-1}$</td>
<td>$6.008 \times 10^{-2}$</td>
<td>$5.838 \times 10^{-3}$</td>
<td>$7.553 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

**Fig. 4.1.** Initial value problem (4.1). $T = 5$. Dots—numerical results, solid line—exact solution. (a) Solutions for $\Delta x_A = 0.2$. (b) Solutions for $\Delta x_A = 0.02$. (c) History of conservation error $S_{1\Delta}(0,t)$ for (a). (d) History of conservation error $S_{1\Delta}(0,t)$ for (b).

Convergence is about of the order 1 in $L^1$-norm and it is even no less than that when (2.3) is adopted.

Nonconservative interface schemes are often applied with good results on Chimera grids, even if discontinuities propagate across grid interfaces. Here we show the computed results of an inviscid flow field developed by a shock moving through a square grid $G_B$ with length 0.5. The square grid intersects in an arbitrary way with an underlying square grid $G_A$ with length 1. Mesh arrangements are shown in Figures 4.3(a) and 4.3(c). Interface nodes of $G_B$ are on its four sides, while those of $G_A$ are the nodes marked by small squares. Conditions at the interface nodes of each grid are given by the bilinear formulas

$V_{i,j}^{n+1} = a x y + b x + c y + d, \ i, j \subset \{i_{AI}, j_{AI}\}$,

$V_{i,j}^{n+1} = a \xi \eta + b \xi + c \eta + d, \ i, j \subset \{i_{BI}, j_{BI}\}$,

where the coefficients for one node of a grid are determined by the numerical solution at the other grid’s nodes surrounding the node. Results in Figure 4.3(b) show that
Table 4.2

Convergence tests of the Riemann problem.

<table>
<thead>
<tr>
<th>$\Delta x_A$, $\Delta x_B$</th>
<th>$|S_{1\Delta}(0,T)|_\infty$</th>
<th>$\sum_m |U_{m}^n(x,T) - U_m^*(x,T)|_{12}^{1/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1, 0.05</td>
<td>$1.657 \times 10^{-1}$</td>
<td>2.412</td>
</tr>
<tr>
<td>0.01, 0.005</td>
<td>$5.655 \times 10^{-3}$</td>
<td>3.032</td>
</tr>
<tr>
<td>0.001, 0.0005</td>
<td>$5.045 \times 10^{-4}$</td>
<td>3.244</td>
</tr>
</tbody>
</table>

Fig. 4.2. Riemann problem for Euler equations. Initially when $x < 5$, $\rho = 1$, $u = 1$, $p = 1$, and when $x > 5$, $\rho = 1$, $u = -1$, $p = 1$. $T = 2.5$. Here, $\rho$, $u$, and $p$ are density, velocity, and pressure, respectively. $\Delta x_A = 0.1$. Dots—numerical solutions, solid line—exact solutions. (a) (2.3) is used. (b) (2.11) and (4.2) are used.

the shock is a little distorted, and the speed of the numerical shock is about 8 percent faster than that of the exact shock, whose location can be seen from the boundary of grid $G_A$. However, it seems that condition $A_2$ is not violated. Accordingly, by decreasing mesh size, conservation error is reduced and the solution for the shock gets better (Figure 4.3(d)).

However, one must be careful if stationary discontinuities are caught right at grid interfaces. Consider now a one-dimensional inviscid flow with a steady shock located at an interface. Using nonconservative matching (2.4), a discontinuity is caught at the interface and a shock moves away from there to the right (Figure 4.4(a)). Numerical results show that condition $B_1$ is violated and conservation error $\|S_{1\Delta}(0,t)\|_\infty$ accumulates with respect to time (Figure 4.4(b)). The calculated stationary discontinuity is a wrong solution state, and, as predicted by (2.23), the difference between the calculated and accurate solutions gets larger and larger with respect to time. Mesh refinement tests also show that the numerical stationary discontinuity is always located at the interface and no improvement can be made about the numerical solution.

In numerical simulations, conservation is necessary for correct shock jumps and locations. Complying with the analysis in this paper, the above numerical examples demonstrate that, under condition $A_1$ or $B_1$, condition $A_2$ or $B_2$, and condition (2.38), the conservation error due to a nonconservative interface scheme decreases as the mesh size gets small. Consequently, inaccuracy for shock jumps and locations caused by such a treatment reduces. Therefore, under these conditions, nonconservative interface schemes are applicable to practical calculations, even if discontinuities are present near grid interfaces.
Fig. 4.3. Computed results of flowfield with a shock moving to the right and passing through a square grid. $\rho_l = 3.948$, $u_l = 4.359$, $p_l = 5.005$, $\rho_r = 1.4$, $u_r = 3$, $p_r = 1$, $T = 0.13$. Here subscript $l$ and $r$ indicate left and right side, respectively. (4.3) is adopted at the interface nodes, a scheme based on TVNI [18] is used within each grid, and $CFL = 0.95$. (a) Mesh arrangement. $G_A: 24 \times 24$, $G_B: 18 \times 18$. (b) Pressure contours for (a), $\|S_{2\Delta}(0,T)\|_\infty = 2.4 \times 10^{-3}$. (c) Mesh arrangement. $G_A: 93 \times 93$, $G_B: 69 \times 69$. (d) Pressure contours for (c), $\|S_{2\Delta}(0,T)\|_\infty = 7.1 \times 10^{-5}$.
A steady shock is located at the interface. Initially when $x < 0$, $\rho = 0.25$, $u = 4$, $p = 0.2857142$, and when $x > 0$, $\rho = 1$, $u = 1$, $p = 3.2857142$. $T = 1$. Interface scheme is (2.4). The Lax–Wendroff scheme is adopted in both subdomains $A$ and $B$, $\Delta x_A = 0.025$, $\Delta x_B = 0.05$, CFL = 0.95.

(a) Pressure solutions. Dots—numerical results, solid line—exact solution. (b) History of conservation error $\|S_1(0,t)\|_{\infty}$.

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REFERENCES


